Unit 9

Differentiating scalar and vector fields

Introduction

This unit investigates fields and explains what can be learned by differentiating them.

Roughly speaking, a **field** is a physical quantity that has definite values at points throughout a region of space. For example, at a given instant, each point in a room has a particular temperature. Near a radiator, the temperature may be 40°C, but near an open door it may be only 10°C. We cannot say that the room has a single temperature, but each point in the room does have a definite temperature, and the distribution of temperatures throughout the room is described by a *temperature field*.

If the room is a cuboid of dimensions $a \times b \times c$, we can choose a Cartesian coordinate system with its origin at one corner and its axes running along three adjacent edges of the cuboid. Then each point in the room can be represented by a triplet of coordinates (x, y, z), and the temperature field can be represented by a function

$$T = T(x, y, z) \quad (0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c),$$

where the conditions in parentheses specify the domain of the function, which corresponds to the region inside the room.

A second example is provided by wind velocity in a given region of the atmosphere. This would be of keen interest to anyone living close to the track of a tornado, for example! We focus on a cubic volume that is fixed relative to the ground, with sides of length 1 kilometre, and arrange the axes of a Cartesian coordinate system to run along three adjacent edges of this cube. At a given instant, the wind velocity may vary throughout the cube, but at each point it has a definite velocity, described by a velocity vector $\mathbf{v}$. The distribution of wind velocities within the cube is described by a *velocity field*, and is represented by a function

$$\mathbf{v} = \mathbf{v}(x, y, z) \quad (0 \le x \le 1000,\ 0 \le y \le 1000,\ 0 \le z \le 1000),$$

where x, y and z are the coordinates of a point (measured in metres), and the conditions in parentheses restrict attention to the region inside the cube, which is the domain of the function.

At each point (x, y, z) in its domain, the function $\mathbf{v}(x, y, z)$ specifies a vector – the wind velocity at that point. A velocity vector $\mathbf{v}$ can be written in component form as

$$\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are Cartesian unit vectors, and v_x, v_y and v_z are the corresponding Cartesian components. So the wind velocity field can be written as

$$\mathbf{v}(x, y, z) = v_x(x, y, z)\,\mathbf{i} + v_y(x, y, z)\,\mathbf{j} + v_z(x, y, z)\,\mathbf{k},$$

which involves three functions $v_x(x, y, z)$, $v_y(x, y, z)$ and $v_z(x, y, z)$. Each of these functions has the domain $0 \le x \le 1000$, $0 \le y \le 1000$, $0 \le z \le 1000$, corresponding to the cubic region of interest.

Fields are classified according to the nature of the physical quantity that they describe. In this module we consider two types of field:

- **scalar fields** describe the distribution of a scalar quantity (such as temperature) throughout a region
- **vector fields** describe the distribution of a vector quantity (such as velocity) throughout a region.

Figure 1 Michael Faraday (1791–1867)

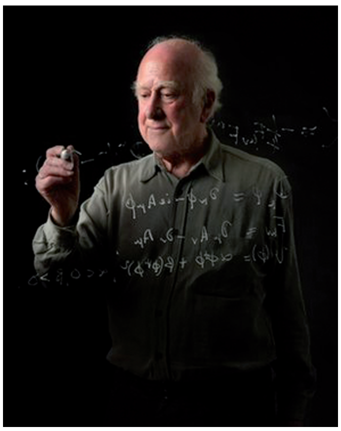


Figure 2 Peter Higgs (1929–)

Fields are everywhere

The example of a temperature field arises naturally in the context of heating a room, and the example of a wind velocity field is clearly important for weather forecasters. However, the full importance of the field concept goes beyond what these examples suggest.

One of the first people to suspect this was Michael Faraday (Figure 1), an extraordinary genius who had only a rudimentary education but gained entry into the scientific world by taking notes in public lectures, presenting a bound copy to the lecturer, and asking if he could help with experiments. As Faraday became more independent, he made revolutionary discoveries about electricity and magnetism.

Mulling over his observations, Faraday became convinced that magnets and electric currents produce magnetic fields in the space around them, and that other magnets and electric currents respond to the magnetic fields that they encounter. Before long, it was realised that electrical and magnetic phenomena were best described using two vector fields: the *electric field* and the *magnetic field*. Faraday was not a mathematician and could not develop his ideas in terms of equations, but he highlighted the importance of the field concept, and the urgent need to develop a *calculus of fields* became clear.

At first, physicists thought that electric and magnetic fields must describe distortions in a mysterious medium, which they called the *ether*. However, this is not the modern view: electric and magnetic fields (along with gravitational fields) are now regarded as part of the fabric of the Universe. Since the 1930s, this view has been extended to matter, as *quantum field theories* treat fundamental particles such as electrons or quarks as states of excitation of various fields. In 1964, Peter Higgs (Figure 2) and others predicted a new type of scalar field called the *Higgs field*. Nearly 50 years later, the existence of this field was confirmed by the Large Hadron Collider near Geneva, and Higgs shared the Nobel Prize for Physics in 2013.

Study guide

This unit is concerned with the description of fields and, in particular, with ways of characterising them by their rates of change with respect to position. These rates of change are expressed in terms of partial

derivatives with respect to the coordinates. You will therefore need to be familiar with partial differentiation, as covered in Unit 7.

One of the themes of the preceding unit on multiple integration was the use of different types of coordinate system. You saw, for example, that a volume integral over a spherical region is simplified by using spherical coordinates. A similar situation applies to fields. Many of the situations considered by scientists involve fields with cylindrical or spherical symmetry, and it is then a great advantage to use cylindrical or spherical coordinates. You will therefore need to be familiar with the coordinate systems introduced in Unit 8: in particular, you will need to be familiar with the concept of a scale factor, as outlined in Section 4 of Unit 8.

The unit is organised as follows. Section 1 gives essential background for the main topics that follow. It defines scalar and vector fields, and describes ways of representing them, both visually and in terms of equations. Section 2 defines the gradient of a scalar field. You met gradients in Unit 7, so part of this section is a review. However, we will go beyond the material in Unit 7 and show how gradients are represented in non-Cartesian coordinate systems. In the process, we will introduce the important concept of the *del operator*, denoted by $\boldsymbol{\nabla}$.

The rest of the unit is concerned with the spatial derivatives of vector fields. In three dimensions, a vector field has three components, each of which may depend on three position coordinates, so there are nine partial derivatives that describe how rapidly a vector field varies with position. Three of these partial derivatives can be grouped together to define a scalar quantity called the *divergence*, and the other six can be grouped together to form a vector quantity called the *curl*. As their names suggest, divergence and curl have direct physical interpretations, which will be explored in this unit and the next. Section 3 discusses divergence, and Section 4 discusses curl. As in the rest of the unit, these quantities will be described in a variety of coordinate systems.

1 Scalar and vector fields

1.1 Preliminary remarks on scalar and vector fields

The Introduction described a field as a physical quantity with definite values at points throughout a region of space. This is a fair description, but some clarification is needed.

- In practice, fields are used to describe physical quantities, and it is generally helpful to keep real examples in mind. Physicists use many different types of field, but in this unit we focus on examples such as temperature and velocity fields that require a minimum of background knowledge.

- Fields are classified according to the type of quantity involved. This module discusses *scalar fields* and *vector fields*. Other types of field exist as well – for example, elastic stress at each point in a solid is best described by a square matrix – but scalar and vector fields are by far the most important in physical applications.
- In this unit we are especially interested in differentiating fields, so we assume that they vary smoothly from point to point. In particular, we assume that all the required partial derivatives exist.

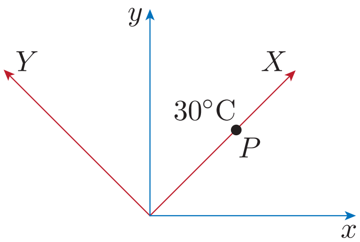


Figure 3 The temperature at P has a definite value, no matter what the orientation of the coordinate system

One other aspect of fields must be discussed. In the case of a *scalar field*, such as temperature, the value of the field at a given point is independent of the orientation of the coordinate system used to label the point. For example, in Figure 3, the blue and red coordinate systems are at 45° to one another. However, the temperature at P is 30°C whether we describe that point by the coordinates $x = 1$, $y = 1$ in the blue coordinate system or by the coordinates $X = \sqrt{2}$, $Y = 0$ in the red coordinate system. Such **invariance in value** is taken as a defining property of a scalar field.

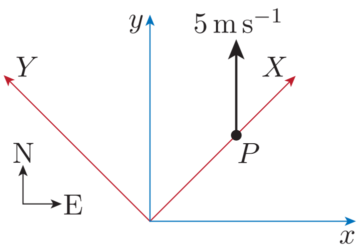


Figure 4 The velocity vector at P has a definite magnitude and direction, but its components depend on the orientation of the coordinate system

In the case of a *vector field*, such as wind velocity, similar ideas apply but the details play out differently. At a given point P, a vector field has a definite magnitude and a definite direction in space, irrespective of the orientation of our coordinate system. For example, in Figure 4 the velocity field at P has a magnitude of 5 metres per second and points in a northward direction. We require that this meaning is preserved no matter what the orientation of the coordinate system. Such **invariance in magnitude and direction** is taken as a defining property of a vector field. Let us explore further what this means.

In Figure 4, the bold arrow indicates the velocity vector at P (5 metres per second in a northward direction). If we describe this vector in the blue coordinate system, we get components $v_x = 0$ and $v_y = 5$ (in metres per second). But if we describe the same vector in the red coordinate system, the components have different values: $v_X = 5/\sqrt{2}$ and $v_Y = 5/\sqrt{2}$. So the components of the vector depend on the orientation of the coordinate system. Nevertheless, these are just different descriptions of the *same* vector – the vector itself does not depend on the orientation of the coordinate system.

Notice, by the way, that it would be incorrect to describe the components of a vector field as scalar fields. This is because the components of a vector field depend on the orientation of the coordinate system but, by definition, scalar fields do not.

In any given coordinate system, fields are described by functions of several variables (the coordinates of points). This unit and the next will describe the differentiation and integration of fields, and you might wonder whether there will be anything new to say, beyond the topics already covered in Units 7 and 8. Indeed there is, and the fundamental reason for this is that scalar and vector fields have properties that transcend the choice of coordinate system. This gives us a richer structure to explore – and one that is directly relevant to descriptions of the physical world.

1.2 Describing scalar fields

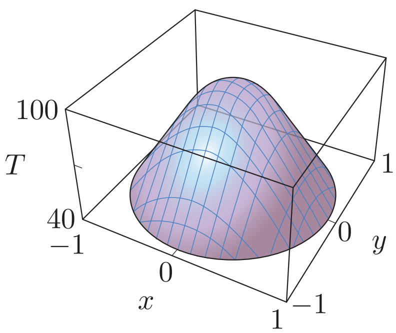
In a given Cartesian coordinate system, a scalar field is described by a single function. For example, a temperature field may be described in two dimensions by a function $T(x, y)$, and in three dimensions by a function $T(x, y, z)$. This idea was discussed in the Introduction of Unit 7, but we give a brief review here.

Suppose that the temperature field on the surface of a circular disc of radius 1 metre is given by the function

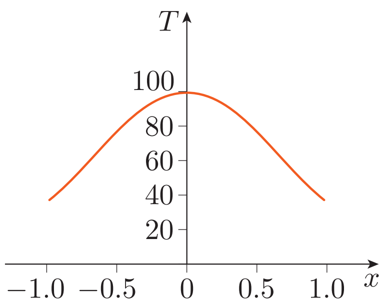
$$T(x, y) = 100\, e^{-(x^2+y^2)} \quad (\sqrt{x^2 + y^2} \leq 1), \tag{1}$$

where T is the temperature in degrees Celsius, x and y are Cartesian coordinates for points on the surface of the disc (measured in metres), and the origin is at the centre of the disc. The condition in parentheses gives the domain of the function, which is the surface of the disc. Clearly, the centre of the disc has temperature $T(0, 0) = 100$, and a point on the edge of the disc has temperature $T(1, 0) = 100\, e^{-1} \simeq 37$ (in degrees Celsius).

There are a number of ways of representing this situation graphically. Figure 5(a) shows a three-dimensional diagram of the temperature on the surface of the disc, with x and y plotted horizontally and T plotted vertically. Figure 5(b) shows a slice through this surface at $y = 0$, which gives a graph of T against x when $y = 0$.



(a)



(b)

Figure 5 Visualisation of the temperature field $T(x, y)$ in equation (1), with values in degrees Celsius: (a) a perspective view with T plotted vertically; (b) a graph of T against x for a slice with $y = 0$

While both of these representations are useful, they are somewhat limited. The perspective view gives a good overall impression, but we cannot read values accurately from its scales. The graph in Figure 5(b) is quantitative, but it does not tell us about the behaviour for other slices (such as a slice with $y = 0.1$).

Perhaps the best tool for visualising a scalar field in two dimensions is to draw a **contour map**. Figure 6 shows a contour map for the temperature field described by equation (1). The orange curves are **contour lines**. Each contour line joins neighbouring points where the temperature has a fixed value, marked next to the contour line.

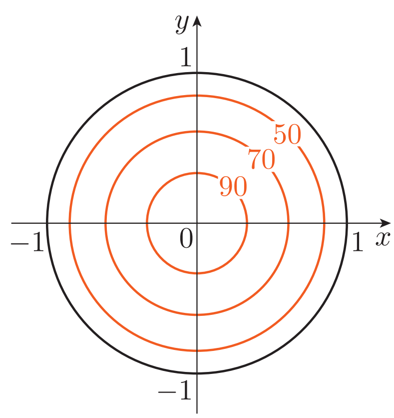


Figure 6 Contour lines (in orange) for equation (1), with values in degrees Celsius

For particular scalar fields, contour lines are sometimes given special names. For example, lines joining points of equal temperature are called *isotherms*, and lines joining points of equal pressure are called *isobars*. We will not bother with these terms here because, from the point of view of mathematics, they are all the same idea.

By studying a contour map, we can get a good idea of the form of a scalar field. In Figure 6, the contours corresponding to higher temperatures are closer to the centre of the disc, so the temperature falls as we move outwards. It is significant that the contour lines are circles around the centre of the disc. This shows that the temperature falls equally in all outward radial directions.

These ideas can be extended to three dimensions. For example, the temperature field throughout a sphere of radius 1 metre could be given by the function

$$T(x,y,z) = 100\, e^{-(x^2+y^2+z^2)} \quad \left(\sqrt{x^2+y^2+z^2} \le 1\right), \tag{2}$$

where T is the temperature in degrees Celsius, x, y and z (in metres) are Cartesian coordinates, and the origin is at the centre of the sphere. The domain of the function is given in parentheses: this is the region occupied by the sphere.

In this case, rather than contour lines, there are **contour surfaces** joining neighbouring points where the temperature has a fixed value. A series of contour surfaces can be imagined at equally-spaced values of temperature. For the field specified in equation (2), these would be concentric spherical surfaces centred on the origin, and a cross-section through these surfaces at $z = 0$ would look exactly like Figure 6. We inevitably struggle to give an accurate impression of contour surfaces using a two-dimensional sketch, and it is generally necessary to show a cross-sectional view.

Scalar fields in polar coordinates

One of the themes that emerged from our study of multiple integrals was the importance of choosing a suitable coordinate system. Apart from Cartesian coordinates, three coordinate systems are of special importance in this module (and in physics and applied mathematics more generally). They are *polar coordinates*, *cylindrical coordinates* and *spherical coordinates*. It is a straightforward task to represent scalar fields in terms of these coordinates. We begin with polar coordinates.

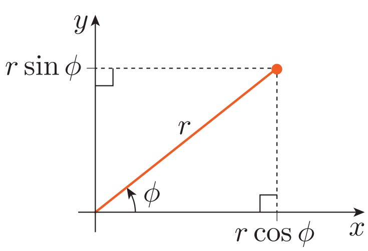


Figure 7 Polar coordinates

As shown in Figure 7, points in the xy-plane can be labelled by **polar coordinates** (r, ϕ), which are related to Cartesian coordinates (x, y) by the equations

$$x = r\cos\phi, \quad y = r\sin\phi. \tag{3}$$

If a two-dimensional scalar field is expressed in Cartesian coordinates as $V(x, y)$, it is easy to express it in polar coordinates: we just replace x and y by r and ϕ using equations (3). This substitution ensures that the scalar field has a definite value at a given point (regardless of the coordinate system used to label it).

For example, to express the two-dimensional temperature field of equation (1) in polar coordinates, we note that

$$\begin{aligned} x^2 + y^2 &= (r\cos\phi)^2 + (r\sin\phi)^2 \\ &= r^2(\cos^2\phi + \sin^2\phi) \\ &= r^2, \end{aligned}$$

The relationship $x^2 + y^2 = r^2$ is worth remembering; it is a consequence of Pythagoras's theorem in Figure 7.

so

$$T(r, \phi) = 100\, e^{-r^2} \quad (r \leq 1). \tag{4}$$

Notice that we have used the symbols $T(x, y)$ and $T(r, \phi)$ to represent the temperature field in Cartesian and polar coordinates, even through these are different mathematical functions. The arguments of the function – (x, y) or (r, ϕ) – indicate whether we intend the function of equation (1) or the function of equation (4). This follows our usual convention, used many times previously in this module, and for good reason: it would be impractical to invent new symbols for temperature (or any other physical quantity) every time we change coordinates.

Notice also that equation (4) is simpler than equation (1) because it depends on just one variable, r. This is an important point. A two-dimensional scalar field that is unchanged by any rotation around the origin is said to be **rotationally symmetric**. It is generally wise to describe such a field in polar coordinates.

Exercise 1

Each of the following expressions represents a two-dimensional scalar field expressed in Cartesian coordinates. Find expressions for these fields in polar coordinates.

(a) $U(x, y) = x^2 - y^2$

(b) $V(x, y) = 2xy$

(c) $W(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}} \; ((x, y) \neq (0, 0))$

Scalar fields in cylindrical coordinates

For three-dimensional fields, the most commonly used non-Cartesian coordinate systems are cylindrical and spherical. Figure 8(a) illustrates a **cylindrical coordinate system** (r, ϕ, z).

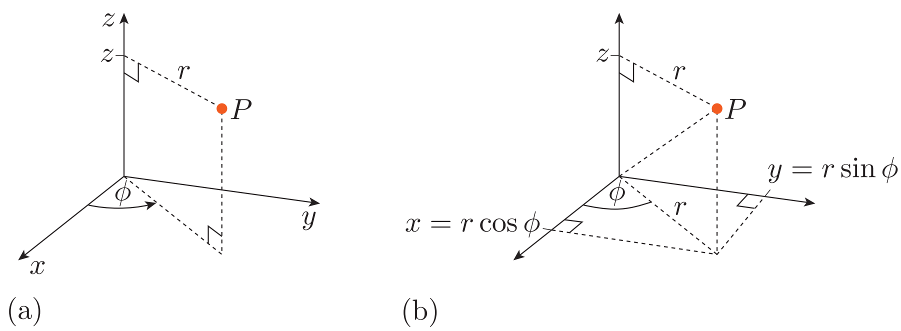


Figure 8 A cylindrical coordinate system: (a) the coordinates (r, ϕ, z); (b) relationship to Cartesian coordinates

The relationship with Cartesian coordinates can be found using trigonometry in Figure 8(b). We get

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z, \tag{5}$$

where

Recall: in cylindrical coordinates, r is not the distance from the origin.

$$r = \sqrt{x^2 + y^2} \tag{6}$$

is the distance from the z-axis. The first two equations in (5) are the same as for two-dimensional polar coordinates, while the third equation, $z = z$, shows that the z-coordinates of cylindrical and Cartesian coordinates are identical.

If a scalar field is a known function of x, y and z, we can use equations (5) to express it in terms of cylindrical coordinates. For example, the scalar field

$$U(x, y, z) = x^2 + y^2 + 2z^2$$

is expressed in Cartesian coordinates. In cylindrical coordinates, it is

$$\begin{aligned} U(r, \phi, z) &= r^2\cos^2\phi + r^2\sin^2\phi + 2z^2 \\ &= r^2 + 2z^2, \end{aligned}$$

and this is a simpler description because it depends on two coordinates rather than three. Any three-dimensional scalar field that is independent of the ϕ-coordinate of cylindrical coordinates is said to be **axially symmetric**. It is usually better to describe such fields in cylindrical coordinates rather than Cartesian coordinates.

Exercise 2

A temperature field (in degrees Celsius) is specified in Cartesian coordinates by

$$T(x, y, z) = 100\, e^{-(x^2+y^2+z^2)} \quad (x^2 + y^2 + z^2 \le 1).$$

Express this field in cylindrical coordinates, and find the temperature at a point with cylindrical coordinates $(r, \phi, z) = (0.5, 0, 0.5)$.

Scalar fields in spherical coordinates

Figure 9(a) shows a **spherical coordinate system** (r, θ, ϕ).

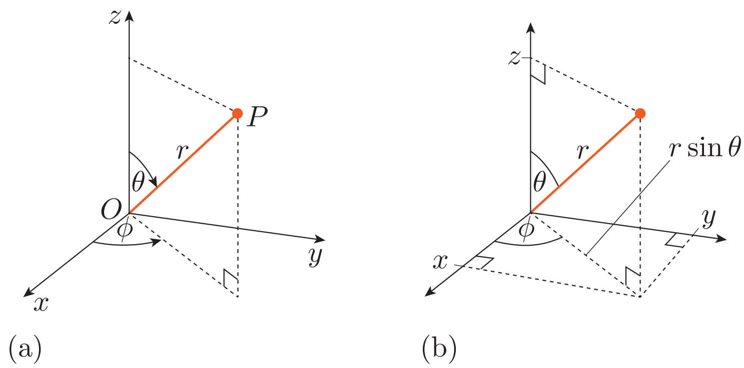


Figure 9 A spherical coordinate system: (a) the coordinates (r, θ, ϕ); (b) relationship to Cartesian coordinates

The relationship with Cartesian coordinates can be found using trigonometry in the right-angled triangles shown in Figure 9(b). We get

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \tag{7}$$

where

$$r = \sqrt{x^2 + y^2 + z^2} \tag{8}$$

is the distance from the origin. We can check that this is consistent with the sum of the squares of x, y and z in equations (7):

$$\begin{aligned} x^2 + y^2 + z^2 &= r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta \\ &= r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) + r^2\cos^2\theta \\ &= r^2(\sin^2\theta + \cos^2\theta) \\ &= r^2. \end{aligned}$$

Using this expression, the three-dimensional temperature field in equation (2) and Exercise 2 can be expressed in spherical coordinates as

$$T(r, \theta, \phi) = 100\, e^{-r^2} \quad (r \leq 1).$$

This is simpler than the Cartesian or cylindrical descriptions because it depends on one coordinate rather than three or two. Any three-dimensional scalar field that is independent of the θ- and ϕ-coordinates of spherical coordinates is said to be **spherically symmetric**. It is generally advisable to describe such fields in spherical coordinates.

Exercise 3

Express the scalar field

$$U(x,y,z) = \frac{z}{(x^2+y^2+z^2)^{1/2}}$$

in terms of:

(a) cylindrical coordinates,

(b) spherical coordinates.

1.3 Describing vector fields

You have seen that a vector field is a function that associates a vector with each point in a given region. For example, if the surface of a river is treated as being flat, the velocity of water flowing on this surface can be represented by a function of the form

$$\mathbf{v}(x,y) = v_x(x,y)\,\mathbf{i} + v_y(x,y)\,\mathbf{j},$$

where the Cartesian coordinates x and y label points on the river's surface, and the unit vectors $\mathbf{i}$ and $\mathbf{j}$ point in the directions of increasing x and increasing y. The domain of $\mathbf{v}(x,y)$ is the river's surface, and within this domain, the functions $v_x(x,y)$ and $v_y(x,y)$ give the x- and y-components of the water's velocity at any point (x,y). This is a two-dimensional vector field.

To take a specific case, suppose that a river has a straight stretch between $x=0$ and $x=100$, with its banks at $y=0$ and $y=10$ (all measured in metres). Then the flow of water (in metres per second) might be described by the function

$$\mathbf{v}(x,y) = \tfrac{1}{10}\,y(10-y)\,\mathbf{i} \quad (0 \le x \le 100,\ 0 \le y \le 10). \tag{9}$$

In this simple model, all the water flows in the direction of $\mathbf{i}$ (the x-direction). The rate of flow does not depend on the downstream distance x, but is fastest in the middle of the river ($y=5$), and drops to zero at either bank (at $y=0$ and $y=10$).

A good way of visualising a two-dimensional vector field is to draw an **arrow map**. For a vector field in the xy-plane, this is done by choosing a selection of points in the xy-plane and drawing an arrow at each point. Each arrow points in the direction of the vector field, and has a length proportional to the magnitude of the field. For example, the arrow map in Figure 10 illustrates the velocity vector field in equation (9). The arrows all point in the x-direction (downstream) and are longer in the middle of the river where the flow is fastest.

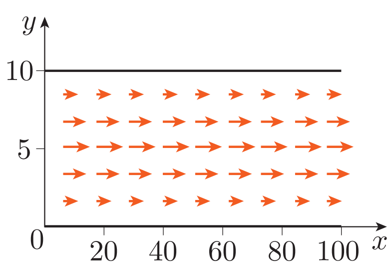


Figure 10 An arrow map for the velocity vector field in equation (9)

Another example of a vector field is provided by the **gravitational field** around a star. The gravitational field at a point P is defined to be the *gravitational force per unit mass* experienced by a small body placed at P. Gravity is an attractive force, and the gravitational field due to the star

points inwards, towards the centre of the star. The gravitational influence of the star decreases as we move away from it, and the magnitude of the gravitational field outside the star turns out to be proportional to $1/r^2$, where r is the distance from the centre of the star.

This is Newton's celebrated inverse square law of gravity.

The gravitational field of a star is three-dimensional, so it cannot be captured on a two-dimensional arrow map. But we can show a cross-section of the field in the xy-plane, as in Figure 11. As you would expect, all the arrows point towards the centre of the star, and their length increases they get closer to the star.

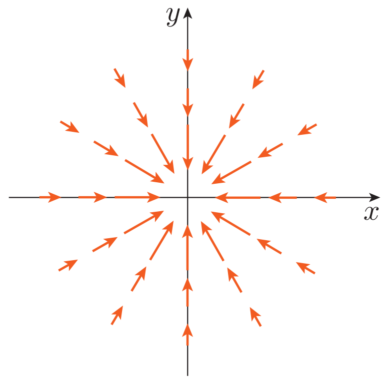


Figure 11 An arrow map for the gravitational field in the xy-plane due to a star at the origin

An alternative graphical way of depicting vector fields is to sketch a **field line map**. Instead of drawing arrows at a set of discrete points, we draw continuous directed lines called **field lines**. At each point along its path, the direction of a field line is the direction of the vector field, and this is indicated by placing one or more arrows on the field line. Vector fields generally vary smoothly in space, so the field lines are generally continuous curves. They tell us the direction of the vector field but they do not, by themselves, reveal the relative magnitudes of the field at different points. Figure 12(a) shows field lines for the velocity vector field of equation (9), and Figure 12(b) shows the field lines for the gravitational field around a star.

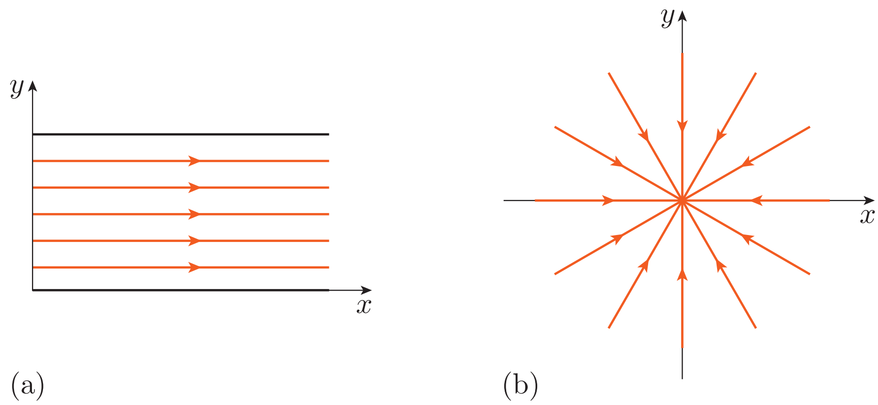


Figure 12 Field line maps for vector fields: (a) the velocity vector field of equation (9); (b) the gravitational field due to a star

So far, we have described vector fields in Cartesian coordinates and Cartesian unit vectors. But for vector fields with axial or spherical symmetry, it is often better to use non-Cartesian coordinates and non-Cartesian unit vectors. We briefly survey how these descriptions work in polar, cylindrical and spherical coordinates before looking more closely at the details.

Vector fields in polar coordinates

When a two-dimensional vector field $\mathbf{v}$ is described in a given Cartesian coordinate system, it takes the form

$$\mathbf{v}(x,y) = v_x(x,y)\,\mathbf{i} + v_y(x,y)\,\mathbf{j}.$$

It is worth noting that the unit vectors **i** and **j** are an important part of this description – as essential as the components v_x and v_y. If we were to choose a different set of Cartesian unit vectors, pointing in different directions, we would have a different set of components.

When we represent a vector field in a given coordinate system, the first step is to define suitable unit vectors. Let us recall how this is done in a two-dimensional Cartesian coordinate system. In this case, the unit vector **i** is a vector of unit magnitude that points in the direction of increasing x, with y held constant. Similarly, the unit vector **j** is a vector of unit magnitude that points in the direction of increasing y, with x held constant. These unit vectors point in fixed directions, perpendicular to one another.

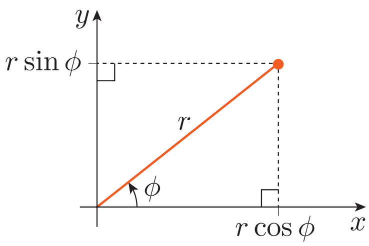


Figure 13 The polar coordinate system

Now consider polar coordinates (r, ϕ) in a plane (shown again in Figure 13). These are related to Cartesian coordinates by the equations

$$x = r\cos\phi, \quad y = r\sin\phi. \tag{10}$$

Based on these coordinates, at any given point P, we can introduce two unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\phi$, as shown in Figure 14.

- $\mathbf{e}_r$ is a vector of unit magnitude pointing in the direction of increasing r, with ϕ held constant.
- $\mathbf{e}_\phi$ is a vector of unit magnitude pointing in the direction of increasing ϕ, with r held constant.

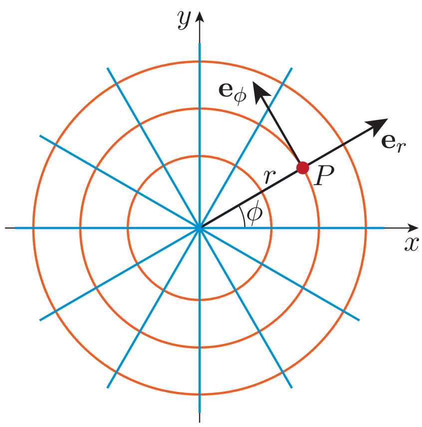


Figure 14 Unit vectors and coordinate lines in polar coordinates: r-coordinate lines in blue, ϕ-coordinate lines in orange

These unit vectors are related to the coordinate lines marked in Figure 14. The r-coordinate lines (in blue) are radial paths along which r increases while ϕ remains constant, while the ϕ-coordinate lines (in orange) are circular paths along which ϕ increases while r remains constant. At any point P, $\mathbf{e}_r$ is tangential to the r-coordinate line through P, and $\mathbf{e}_\phi$ is tangential to the ϕ-coordinate line through P. Because the radial paths meet the circles at right angles, the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\phi$ are mutually orthogonal.

Using trigonometry in Figure 15, we can resolve $\mathbf{e}_r$ and $\mathbf{e}_\phi$ along the directions $\mathbf{i}$ and $\mathbf{j}$ to get the following useful formulas.

$$\left.\begin{aligned}\mathbf{e}_r &= \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j},\\ \mathbf{e}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}.\end{aligned}\right\} \tag{11}$$

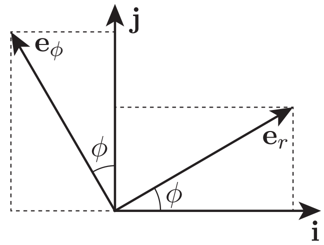


Figure 15 Relating polar unit vectors to Cartesian unit vectors

Exercise 4

Show that the vectors given in equations (11) are unit vectors that are orthogonal to one another.

The Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ are constant vectors – they remain the same, no matter which point (x, y) is being described. But the same is not true for the polar unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\phi$. This is implicit in equations (11) and is illustrated in Figure 16. At point P, the radial unit vector $\mathbf{e}_r$ points directly away from the origin in the direction OP; at another point Q, it points in the direction OQ. The 'transverse' unit vector $\mathbf{e}_\phi$, which is always perpendicular to $\mathbf{e}_r$, also varies with position.

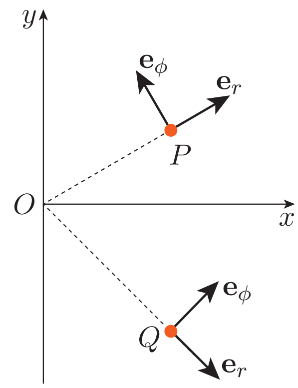


Figure 16 The polar unit vectors vary from one point to another

At first sight, this may seem to be an unwelcome complication, but in many cases it gives us the freedom to replace a complicated description in Cartesian coordinates by a simpler description in polar coordinates. The vectors $\mathbf{e}_r$ and $\mathbf{e}_\phi$ tell us the *local* radial and transverse directions at a given point, and these are sometimes very significant. For example, a mass at the origin produces an inward radial gravitational field, which is best described using radial unit vectors – this is more natural and simpler than introducing Cartesian unit vectors, which point in arbitrary directions unrelated to the direction of the gravitational field.

An example of a two-dimensional vector field expressed in polar coordinates is provided by

$$\mathbf{v}(r, \phi) = 2r\,\mathbf{e}_r. \tag{12}$$

At each point, this field points radially outwards, and its magnitude is proportional to the distance from the origin. The corresponding arrow map is shown in Figure 17, where you should note that the arrows point radially away from the origin, and have a length that increases steadily as we move further away from the origin.

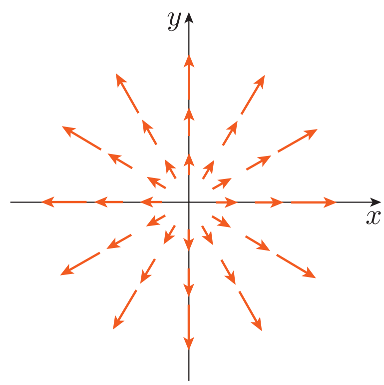


Figure 17 An arrow map for the vector field in equation (12)

The vector field given in equation (12) can be converted to Cartesian coordinates by using equations (10) and (11). At a point with coordinates (x, y) we get

$$\mathbf{v}(x, y) = 2r(\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}) = 2x\,\mathbf{i} + 2y\,\mathbf{j}.$$

This description is entirely equivalent to equation (12), but the link to Figure 17 is a little less obvious. In this case, polar coordinates provide the simplest and most transparent description of the field.

Vector fields in cylindrical coordinates

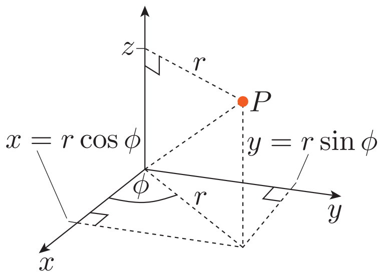


Figure 18 The cylindrical coordinate system

Three-dimensional vector fields can also be described in non-Cartesian coordinates. Cylindrical coordinates (r, ϕ, z) are shown again in Figure 18. They are related to Cartesian coordinates by the equations

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z, \tag{13}$$

where $r = \sqrt{x^2 + y^2}$ is the distance from the z-axis (*not* the distance from the origin) in this coordinate system.

At any given point P, we can introduce three unit vectors $\mathbf{e}_r$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$, based on cylindrical coordinates. These are illustrated in Figure 19 and defined as follows.

- $\mathbf{e}_r$ points in the direction of increasing r, with ϕ and z held constant.
- $\mathbf{e}_\phi$ points in the direction of increasing ϕ, with r and z held constant.
- $\mathbf{e}_z$ points in the direction of increasing z, with r and ϕ held constant.

Each of these unit vectors is tangential to a particular coordinate line along which just one of the cylindrical coordinates (r, ϕ or z) increases while the other two remain fixed.

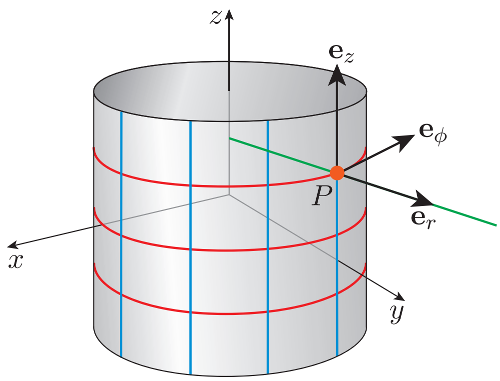


Figure 19 Unit vectors and coordinate lines in cylindrical coordinates: r-coordinate lines are shown in green, ϕ-coordinate lines in red and z-coordinate lines in blue

The unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\phi$ lie in a plane parallel to the xy-plane and are identical to the polar unit vectors, while the unit vector $\mathbf{e}_z$ points in the z-direction and is identical to the unit vector $\mathbf{k}$ of Cartesian coordinates. Hence equations (11) still apply for $\mathbf{e}_r$ and $\mathbf{e}_\phi$, and we have the following results.

$$\left.\begin{aligned} \mathbf{e}_r &= \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}, \\ \mathbf{e}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}, \\ \mathbf{e}_z &= \mathbf{k}. \end{aligned}\right\} \tag{14}$$

These equations show that the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\phi$ depend on position, but $\mathbf{e}_z$ remains constant, being equal to the Cartesian unit vector $\mathbf{k}$.

An example of a vector field expressed in cylindrical coordinates is given by

$$\mathbf{v}(r, \phi, z) = r\,\mathbf{e}_\phi. \tag{15}$$

The field lines for this field are circles around the z-axis, as shown in Figure 20.

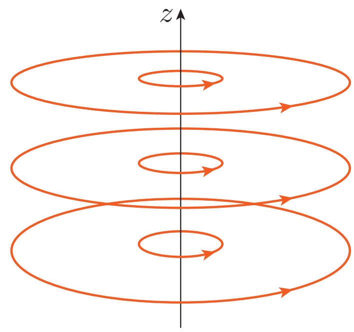


Figure 20 Field lines for the vector field in equation (15)

Vector fields in spherical coordinates

Spherical coordinates (r, θ, ϕ) are shown again in Figure 21. They are related to Cartesian coordinates by the equations

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \tag{16}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from the origin.

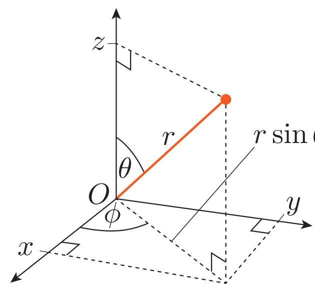


Figure 21 The spherical coordinate system

At any given point, we can introduce three unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$. These are illustrated in Figure 22 and defined as follows.

- $\mathbf{e}_r$ points in the direction of increasing r, with θ and ϕ held constant.
- $\mathbf{e}_\theta$ points in the direction of increasing θ, with r and ϕ held constant.
- $\mathbf{e}_\phi$ points in the direction of increasing ϕ, with r and θ held constant.

Each of these unit vectors is tangential to a coordinate line along which just one of the coordinates (r, θ or ϕ) increases while the other two remain fixed.

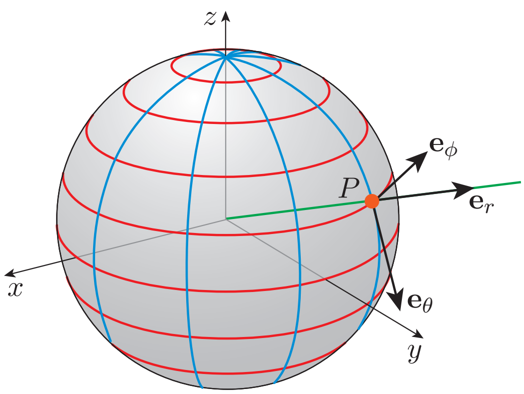


Figure 22 Unit vectors and coordinate lines in spherical coordinates: r-coordinate lines are shown in green, θ-coordinate lines in blue and ϕ-coordinate lines in red

The spherical unit vectors can be visualised as follows. Imagine a spherical coordinate system with its origin at the Earth's centre, and a positive z-axis that points from the origin through the North Pole. Then at a typical point on the Earth's surface, the vector $\mathbf{e}_r$ points vertically upwards, the vector $\mathbf{e}_\theta$ points southwards, and the vector $\mathbf{e}_\phi$ points eastwards. Obviously, these three unit vectors are mutually orthogonal.

There is no unique southward direction at the North or South Pole, but such isolated exceptions can be safely ignored in this module.

Note carefully that $\mathbf{e}_r$ points radially away from the origin in spherical coordinates. This is not the same as the direction of $\mathbf{e}_r$ in cylindrical coordinates (which points radially away from the z-axis). For this reason, it is always important to state clearly which coordinate system is being used. You cannot assume that the symbols speak for themselves.

While the meanings of the spherical unit vectors are clear, the three-dimensional trigonometry needed to express $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ in terms of the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ is rather cumbersome. For the moment, we just quote the results.

$$\left.\begin{aligned}\mathbf{e}_r &= \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k},\\ \mathbf{e}_\theta &= \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k},\\ \mathbf{e}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}.\end{aligned}\right\} \tag{17}$$

You should take these equations on trust for the moment; in Subsection 1.5 we will derive them using an alternative (non-geometric) route.

An example of a vector field that is conveniently described in spherical coordinates is the gravitational field around a star. With a spherical coordinate system centred on the star, this field is

$\mathbf{g}$ is the gravitational force per unit mass experienced by a body in the vicinity of the star.

$$\mathbf{g}(r,\theta,\phi) = -\frac{GM}{r^2}\,\mathbf{e}_r \quad (r > 0), \tag{18}$$

where G is a positive constant (called the constant of gravitation) and M is the mass of the star.

Since M, G and r are all positive, the minus sign on the right-hand side of equation (18) implies that the gravitational field vector $\mathbf{g}$ points in the opposite direction to $\mathbf{e}_r$. In other words, it points radially inwards, *towards* the star, corresponding to gravitational *attraction*. You can also see from equation (18) that the magnitude of $\mathbf{g}$ is independent of θ and ϕ, and decreases as $1/r^2$ as the distance r from the star increases. The ability to read the meaning of equations in this way is a valuable skill for scientists, engineers and all who need to relate equations to the real world.

1.4 Vector field conversions

In principle, you are free to choose whichever coordinate system you like, but the physical situation often singles out a preferred coordinate system – one that makes calculations easier. For all their apparent simplicity, Cartesian coordinates are not always the best choice. For example, the gravitational field of a star has spherical symmetry, and is best described in spherical coordinates, as in equation (18).

We may be given a vector field in one coordinate system, and wish to express it in another coordinate system.

As an example, suppose that we are given a vector field

$$\mathbf{v}(x,y) = v_x(x,y)\,\mathbf{i} + v_y(x,y)\,\mathbf{j} \tag{19}$$

in Cartesian coordinates, and we want to express it in polar coordinates, that is, in the form

$$\mathbf{v}(r,\phi) = v_r(r,\phi)\,\mathbf{e}_r + v_\phi(r,\phi)\,\mathbf{e}_\phi. \tag{20}$$

We need to find the functions $v_r(r,\phi)$ and $v_\phi(r,\phi)$ that describe how the polar components of the given field vary from point to point.

A vector field does not depend on the orientation of the coordinate system used to describe it. So if a particular point has Cartesian coordinates (x,y) and polar coordinates (r,ϕ), we must have

$$\mathbf{v}(x,y) = \mathbf{v}(r,\phi). \tag{21}$$

On the understanding that equations (19) and (20) refer to the same point, we can simplify our notation by omitting the arguments (x,y) and (r,ϕ). We write

$$\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j}, \tag{22}$$

$$\mathbf{v} = v_r\,\mathbf{e}_r + v_\phi\,\mathbf{e}_\phi. \tag{23}$$

Both of these equations refer to the *same* vector $\mathbf{v}$, expressed in different coordinate systems.

The key to finding an expression for v_r is to note that $\mathbf{e}_r$ and $\mathbf{e}_\phi$ are orthogonal and of unit magnitude, so $\mathbf{e}_r \cdot \mathbf{e}_\phi = 0$ and $\mathbf{e}_r \cdot \mathbf{e}_\mathbf{r} = 1$. Taking the scalar product of both sides of equation (23) with $\mathbf{e}_r$ then gives

$$\mathbf{e}_r \cdot \mathbf{v} = v_r\,\mathbf{e}_r \cdot \mathbf{e}_r + v_\phi\,\mathbf{e}_r \cdot \mathbf{e}_\phi = v_r.$$

So

$$v_r = \mathbf{e}_r \cdot \mathbf{v}.$$

The scalar product on the right-hand side of this equation can be expressed in Cartesian coordinates by taking $\mathbf{e}_r$ from equations (11) and $\mathbf{v}$ from equation (22). We get

$$v_r = (\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}) \cdot (v_x\,\mathbf{i} + v_y\,\mathbf{j}) = \cos\phi\,v_x + \sin\phi\,v_y.$$

An expression for v_y can be found in a similar way. In this case, taking the scalar product of both sides of equation (23) with $\mathbf{e}_\phi$ gives

$$\mathbf{e}_\phi \cdot \mathbf{v} = v_r\,\mathbf{e}_\phi \cdot \mathbf{e}_r + v_\phi\,\mathbf{e}_\phi \cdot \mathbf{e}_\phi = v_\phi.$$

So

$$v_\phi = \mathbf{e}_\phi \cdot \mathbf{v}.$$

Using equations (11) and (22) to expand the scalar product then gives

$$v_\phi = (-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}) \cdot (v_x\,\mathbf{i} + v_y\,\mathbf{j}) = -\sin\phi\,v_x + \cos\phi\,v_y.$$

Example 1

A field like this could describe the velocity at points on a steadily rotating turntable.

A two-dimensional vector field is expressed in Cartesian coordinates as

$$\mathbf{v}(x,y) = -y\,\mathbf{i} + x\,\mathbf{j}.$$

Express this field in polar coordinates (r,ϕ).

Solution

In polar coordinates, the field is expressed as

$$\mathbf{v}(r,\phi) = v_r\,\mathbf{e}_r + v_\phi\,\mathbf{e}_\phi.$$

We have

$$\begin{aligned} v_r &= \mathbf{e}_r \cdot \mathbf{v} \\ &= (\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}) \cdot (-y\,\mathbf{i} + x\,\mathbf{j}) \\ &= -y\cos\phi + x\sin\phi. \end{aligned}$$

Similarly,

$$\begin{aligned} v_\phi &= \mathbf{e}_\phi \cdot \mathbf{v} \\ &= (-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}) \cdot (-y\,\mathbf{i} + x\,\mathbf{j}) \\ &= y\sin\phi + x\cos\phi. \end{aligned}$$

The coordinate transformation equations for polar coordinates are $x = r\cos\phi$ and $y = r\sin\phi$. Using these, we get

$$\begin{aligned} v_r &= -(r\sin\phi)\cos\phi + (r\cos\phi)\sin\phi = 0, \\ v_\phi &= (r\sin\phi)\sin\phi + (r\cos\phi)\cos\phi = r(\sin^2\phi + \cos^2\phi) = r, \end{aligned}$$

So in polar coordinates,

$$\mathbf{v}(r,\phi) = r\,\mathbf{e}_\phi.$$

The method that we have just used to convert a vector field expressed in Cartesian form into polar coordinates can be extended. It relies on the fact that the polar unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\phi$ are orthogonal. A similar method works in cylindrical and spherical coordinates, and all other orthogonal coordinate systems.

To take a general three-dimensional case, suppose that we are given a vector field $\mathbf{F}$ in Cartesian coordinates,

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k},$$

and we wish to express it in an orthogonal coordinate system (u,v,w), with orthogonal unit vectors $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_w$. Then we write

$$\mathbf{F} = F_u\,\mathbf{e}_u + F_v\,\mathbf{e}_v + F_w\,\mathbf{e}_w,$$

where the components F_u, F_v and F_w are unknown functions of (u,v,w).

These functions can be found using the following procedure.

Procedure 1 Finding components in orthogonal coordinates

For a vector field $\mathbf{F}$ in an orthogonal coordinate system (u, v, w), the component F_u in the direction of $\mathbf{e}_u$ is found as follows.

1. Write down

$$F_u = \mathbf{e}_u \cdot \mathbf{F},$$

and expand the scalar product on the right-hand side using Cartesian expressions for $\mathbf{e}_u$ and $\mathbf{F}$ (involving $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$).

2. The resulting expression generally depends on (x, y, z). Use coordinate transformation equations of the form

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w),$$

to obtain an expression for F_u solely in terms of u, v and w.

The following example illustrates how this procedure is used.

Example 2

In Cartesian coordinates, a vector field takes the form

$$\mathbf{B}(x, y, z) = yz\,\mathbf{i} - xz\,\mathbf{j} + 2xy\,\mathbf{k}.$$

Express this field in cylindrical coordinates (r, ϕ, z).

Solution

We write the field as

$$\mathbf{B}(r, \phi, z) = B_r\,\mathbf{e}_r + B_\phi\,\mathbf{e}_\phi + B_z\,\mathbf{e}_z,$$

where the cylindrical unit vectors are given by equations (14):

$$\mathbf{e}_r = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}, \quad \mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}, \quad \mathbf{e}_z = \mathbf{k}.$$

Using Procedure 1, we get

$$\begin{aligned} B_r &= \mathbf{e}_r \cdot \mathbf{B} = (\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}) \cdot (yz\,\mathbf{i} - xz\,\mathbf{j} + 2xy\,\mathbf{k}) \\ &= yz\cos\phi - xz\sin\phi, \\ B_\phi &= \mathbf{e}_\phi \cdot \mathbf{B} = (-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}) \cdot (yz\,\mathbf{i} - xz\,\mathbf{j} + 2xy\,\mathbf{k}) \\ &= -yz\sin\phi - xz\cos\phi, \\ B_z &= \mathbf{e}_z \cdot \mathbf{B} = \mathbf{k} \cdot (yz\,\mathbf{i} - xz\,\mathbf{j} + 2xy\,\mathbf{k}) = 2xy. \end{aligned}$$

These expressions still involve (x, y, z), but in cylindrical coordinates

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z,$$

so we get

$$\begin{aligned} B_r &= (r\sin\phi)\,z\cos\phi - (r\cos\phi)\,z\sin\phi = 0, \\ B_\phi &= -(r\sin\phi)\,z\sin\phi - (r\cos\phi)\,z\cos\phi = -rz(\sin^2\phi + \cos^2\phi) = -rz, \\ B_z &= 2r^2\cos\phi\sin\phi = r^2\sin(2\phi). \end{aligned}$$

Hence in cylindrical coordinates,

$$\mathbf{B}(r,\phi,z) = -rz\,\mathbf{e}_\phi + r^2\sin(2\phi)\,\mathbf{e}_z.$$

Exercise 5

In Cartesian coordinates, a vector field takes the form

$$\mathbf{F}(x,y,z) = (x+y)\,\mathbf{i} + (y-x)\,\mathbf{j} + 3z\,\mathbf{k}.$$

Express this field in cylindrical coordinates (r,ϕ,z).

Exercise 6

In Cartesian coordinates, a vector field takes the form

$$\mathbf{F}(x,y,z) = z\,\mathbf{k}.$$

Use equations (17) to express this field in spherical coordinates (r,θ,ϕ).

You have seen how to convert a vector field from Cartesian coordinates to non-Cartesian orthogonal coordinates. We sometimes need the opposite conversion, from non-Cartesian to Cartesian coordinates. For example, a vector field may be given in spherical coordinates as

$$\mathbf{F} = \frac{1}{r^2}\,\mathbf{e}_r.$$

To express this field in Cartesian coordinates, we begin by using equations (17) to express $\mathbf{e}_r$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. This gives

$$\mathbf{F} = \frac{1}{r^2}\,(\sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}).$$

The remaining task is to express the quantities involving r, θ and ϕ in terms of x, y and z. This can often be done by inspection. In the present case, the coordinate transformation equations

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

give

$$\sin\theta\cos\phi = \frac{x}{r}, \quad \sin\theta\sin\phi = \frac{y}{r}, \quad \cos\theta = \frac{z}{r}.$$

Also, it is clear from geometry (and equation (8)) that the distance from the origin is $r = (x^2+y^2+z^2)^{1/2}$. Hence the vector field can be expressed as

$$\begin{aligned}\mathbf{F} &= \frac{1}{r^2}\left(\frac{x}{r}\,\mathbf{i} + \frac{y}{r}\,\mathbf{j} + \frac{z}{r}\,\mathbf{k}\right)\\ &= \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}.\end{aligned}$$

Exercise 7

In spherical coordinates, a vector field takes the form

$$\mathbf{F}(r,\theta,\phi) = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta.$$

Express this field in Cartesian coordinates.

1.5 Unit vectors and scale factors

This subsection develops the theme of scale factors introduced in Unit 8. It shows that unit vectors in all coordinate systems can be obtained from a single formula (equation (27)). This useful formula allows us to avoid elaborate geometric arguments in three dimensions. It will be used later on, but its derivation is not assessed.

To take a unified view, we consider any three-dimensional coordinate system with coordinates (u, v, w). The coordinates u, v and w are related to Cartesian coordinates by equations of the type

$$x = x(u,v,w), \quad y = y(u,v,w), \quad z = z(u,v,w).$$

In spherical coordinates, for example, $(u, v, w) = (r, \theta, \phi)$, and these equations take the form

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

We can define coordinate lines along which just one coordinate increases while the other two remain fixed. Along a u-coordinate line, for example, u increases while v and w have fixed values. Then we can imagine taking a small step along a u-coordinate line. In general, the chain rule tells us that

$$\delta x = \frac{\partial x}{\partial u}\,\delta u + \frac{\partial x}{\partial v}\,\delta v + \frac{\partial x}{\partial w}\,\delta w.$$

But along a u-coordinate line, v and w are held constant, so $\delta v = 0$ and $\delta w = 0$, giving

$$\delta x = \frac{\partial x}{\partial u}\,\delta u.$$

Similarly,

$$\delta y = \frac{\partial y}{\partial u}\,\delta u \quad \text{and} \quad \delta z = \frac{\partial z}{\partial u}\,\delta u.$$

The displacement along the u-coordinate line produced by a small increase in u is therefore given by the vector

Unit 8, Subsection 4.2 gave a similar argument.

$$\delta x\,\mathbf{i} + \delta y\,\mathbf{j} + \delta z\,\mathbf{k} = \left(\frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j} + \frac{\partial z}{\partial u}\,\mathbf{k}\right)\delta u. \tag{24}$$

This vector describes a tiny displacement along the u-coordinate line in the direction of increasing u. The crucial point is that this is just the direction of the required unit vector $\mathbf{e}_u$. The factor δu in equation (24) simply scales the vector in brackets without changing its direction. We can therefore concentrate on the vector in brackets, which is denoted by

$$\mathbf{T}_u = \frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j} + \frac{\partial z}{\partial u}\,\mathbf{k}. \tag{25}$$

Because this vector is tangential to the u-coordinate line, it is called a **tangent vector**.

To obtain the unit vector $\mathbf{e}_u$, we just need to scale $\mathbf{T}_u$ by $1/|\mathbf{T}_u|$. So we get

$$\mathbf{e}_u = \frac{1}{|\mathbf{T}_u|}\,\mathbf{T}_u, \tag{26}$$

where

$$|\mathbf{T}_u| = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}.$$

Referring back to Unit 8, Section 4, you can see that $|\mathbf{T}_u|$ is just the **scale factor** h_u for the u-coordinate. We have therefore derived the following result for any unit vector.

Calculating unit vectors

In any orthogonal coordinate system (u, v, w) with

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w),$$

the unit vector $\mathbf{e}_u$ in the direction of the u-coordinate line is related to the Cartesian unit vectors by

$$\mathbf{e}_u = \frac{1}{h_u}\left(\frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j} + \frac{\partial z}{\partial u}\,\mathbf{k}\right), \tag{27}$$

where h_u is the scale factor

$$h_u = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}. \tag{28}$$

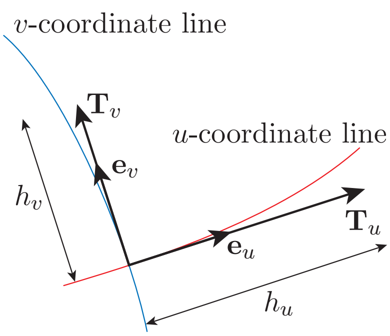


Figure 23 Tangent vectors, unit vectors and scale factors

Of course, similar equations apply for two-dimensional coordinate systems in the xy-plane, but without the terms involving z and $\mathbf{k}$.

The geometric relationship between tangent vectors, unit vectors and scale factors is illustrated in Figure 23.

Example 3

Use tangent vectors to derive expressions for the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\phi$ of polar coordinates.

Solution

In polar coordinates we have $x = r\cos\phi$ and $y = r\sin\phi$. So the tangent vectors in the r- and ϕ-directions are

$$\mathbf{T}_r = \frac{\partial x}{\partial r}\,\mathbf{i} + \frac{\partial y}{\partial r}\,\mathbf{j} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j},$$
$$\mathbf{T}_\phi = \frac{\partial x}{\partial \phi}\,\mathbf{i} + \frac{\partial y}{\partial \phi}\,\mathbf{j} = -r\sin\phi\,\mathbf{i} + r\cos\phi\,\mathbf{j}.$$

These vectors have magnitudes

$$h_r = \sqrt{\cos^2\phi + \sin^2\phi} = 1,$$
$$h_\phi = \sqrt{(-r\sin\phi)^2 + (r\cos\phi)^2} = r,$$

which are the familiar scale factors of polar coordinates. So the unit vectors are

$$\mathbf{e}_r = \frac{1}{h_r}\,\mathbf{T}_r = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j},$$
$$\mathbf{e}_\phi = \frac{1}{h_\phi}\,\mathbf{T}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j},$$

in agreement with equations (11).

Exercise 8

Use tangent vectors to derive expressions for the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ of spherical coordinates.

2 The gradient of a scalar field

We can now discuss the main topic of this unit – describing how rapidly fields change in space. This section describes the spatial rates of change of *scalar fields*, while Sections 3 and 4 describe spatial rates of change of *vector fields*.

The spatial rates of change of a scalar field $V(x, y, z)$ can be described using three partial derivatives: $\partial V/\partial x$, $\partial V/\partial y$ and $\partial V/\partial z$. An important quantity can be constructed from these – the *gradient vector*, which you met in Unit 7. Subsection 2.1 gives a brief review of the gradient vector in Cartesian coordinates, covering much the same ground as in Unit 7. In Subsection 2.2 we go further and explain how the gradient vector is calculated in non-Cartesian coordinates.

2.1 Gradients in Cartesian coordinates

Suppose that $V(x, y, z)$ is a three-dimensional scalar field expressed in Cartesian coordinates. Then you know how to find the gradient of this field. According to Unit 7,

$$\mathbf{grad}\, V = \frac{\partial V}{\partial x}\,\mathbf{i} + \frac{\partial V}{\partial y}\,\mathbf{j} + \frac{\partial V}{\partial z}\,\mathbf{k}. \tag{29}$$

The symbol $\boldsymbol{\nabla}$ is read as 'del' or 'nabla'.

This is often written in a different notation, using the symbol $\boldsymbol{\nabla}$ instead of **grad**. In this notation, the gradient of $V(x, y, z)$ is

$$\boldsymbol{\nabla} V = \frac{\partial V}{\partial x}\,\mathbf{i} + \frac{\partial V}{\partial y}\,\mathbf{j} + \frac{\partial V}{\partial z}\,\mathbf{k}. \tag{30}$$

Because $\boldsymbol{\nabla} V$ is a vector, the symbol $\boldsymbol{\nabla}$ is printed in bold type, and should be underlined in handwriting. We will use the notations $\mathbf{grad}\, V$ and $\boldsymbol{\nabla} V$ interchangeably.

For a two-dimensional scalar field $V(x, y)$, the gradient is

$$\mathbf{grad}\, V = \boldsymbol{\nabla} V = \frac{\partial V}{\partial x}\,\mathbf{i} + \frac{\partial V}{\partial y}\,\mathbf{j}. \tag{31}$$

Example 4

Calculate the gradient of $V(x, y, z) = xy^2z^3$.

Solution

Using equation (29), the gradient is

$$\begin{aligned}\mathbf{grad}\, V &= \frac{\partial V}{\partial x}\,\mathbf{i} + \frac{\partial V}{\partial y}\,\mathbf{j} + \frac{\partial V}{\partial z}\,\mathbf{k}\\ &= y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}.\end{aligned}$$

In the alternative $\boldsymbol{\nabla}$ notation, this is written as

$$\boldsymbol{\nabla} V = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}.$$

Exercise 9

Find the gradients of the following functions.

(a) $V(x, y) = e^{x^2+y^2}$

(b) $V(x, y, z) = e^{x^2+y^2+z^2}$

Properties of gradient

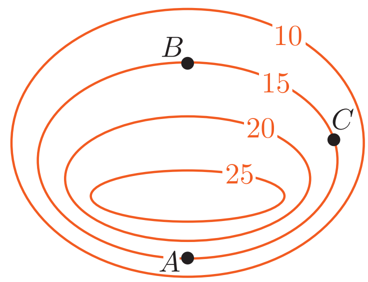
The properties of the gradient vector were outlined in Unit 7, mainly for functions of two variables. Given a function $V(x, y)$, the gradient vector $\mathbf{grad}\, V$ at any given point P is a vector in the xy-plane. This vector satisfies the following properties.

Have another look at Unit 7 if you need to refresh your memory.

- Its direction is that in which V increases most rapidly. This direction is perpendicular to the contour lines of $V(x,y)$.
- Its magnitude is the maximum rate of increase of V with respect to distance travelled in the xy-plane.

Exercise 10

The figure in the margin shows a contour map of a two-dimensional scalar field V. Use arrows to indicate the directions and relative magnitudes of $\mathbf{grad}\, V$ at points A, B and C. You should use the convention that an arrow represents the value of $\mathbf{grad}\, V$ at its own tail. The absolute lengths of the arrows are arbitrary, but you should choose their relative lengths appropriately.



Exercise 11

A two-dimensional scalar field is given by

$$V(x,y) = \ln(\sqrt{x^2+y^2}) \quad ((x,y) \neq (0,0)).$$

Find the gradient of this field, and draw a sketch showing the contour line of V through $(1,1)$ and an arrow representing $\boldsymbol{\nabla} V$ at this point.

Most of the scalar fields met in science vary in three-dimensional space. The properties of gradient in three dimensions are natural extensions of those in two dimensions.

Properties of gradient in three dimensions

Given a function $V(x,y,z)$, at any given point $\mathbf{grad}\, V$ is a vector in three-dimensional space.

- Its direction is that in which V increases most rapidly. This direction is perpendicular to the contour surfaces of V.
- Its magnitude is the maximum rate of increase of V with respect to distance travelled in three-dimensional space.

All these properties of gradient, in two and three dimensions, are taken as known facts in the context of this unit. For *scalar fields*, however, we can take things a step further.

Remember that, by definition, the values of a scalar field V do not depend on the orientation of the coordinate system. It follows that at any given point, the direction and magnitude of steepest increase of V, and hence the direction and magnitude of $\mathbf{grad}\, V$, do not depend on the orientation of the coordinate system. This means that $\mathbf{grad}\, V$ is a vector field in the full sense of the term, as defined in Subsection 1.1.

Gradient of a scalar field

The gradient of a scalar field is a *vector field*: at each point, $\mathbf{grad}\, V$ has a definite magnitude and a definite direction in space that do not depend on the orientation of the coordinate system.

We will use the terms gradient, gradient vector and gradient vector field interchangeably.

This conclusion is a deep one. Suppose that a Cartesian coordinate system has coordinates (x, y, z) and unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, and that it is rotated to create another Cartesian coordinate system with coordinates (X, Y, Z) and unit vectors $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{K}$. Then a scalar field

$$f(x, y, z) = f(X, Y, Z)$$

Following our usual convention, f labels two *different* functions: $f(x, y, z)$ and $f(X, Y, Z)$.

has gradient

$$\mathbf{grad}\, f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k} = \frac{\partial f}{\partial X}\,\mathbf{I} + \frac{\partial f}{\partial Y}\,\mathbf{J} + \frac{\partial f}{\partial Z}\,\mathbf{K}.$$

The equality of the two expressions on the right is not a trivial fact, but is guaranteed because $\mathbf{grad}\, f$ is a *vector field* and therefore cannot depend on the orientation of the coordinate system used to describe it.

Exercise 12

A three-dimensional scalar field is given by

$$V(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \quad ((x, y, z) \neq (0, 0, 0)).$$

Calculate the gradient of this field.

Exercise 13

The temperature (in degrees Celsius) in a certain region of space is given by the scalar field

$$T(x, y, z) = 1000 \exp(-(x^2 + 2y^2 + 2z^2)),$$

where x, y and z are measured in metres.

(a) Calculate the gradient of this scalar field at the point $(1, 1, 1)$.

(b) Specify a unit vector $\widehat{\mathbf{n}}$ that gives the direction of the most rapid increase in temperature on moving away from $(1, 1, 1)$.

Gradients and small changes

The components of $\boldsymbol{\nabla} V$ are $\partial V/\partial x$, $\partial V/\partial y$ and $\partial V/\partial z$, which are the rates of change of V in the x-, y- and z-directions. However, if we know $\boldsymbol{\nabla} V$, we can also deduce how V changes in any other direction.

Suppose that we make a tiny displacement from a point (x, y, z) to a neighbouring point $(x + \delta x, y + \delta y, z + \delta z)$. As a result of this displacement, a scalar field $V(x, y, z)$ changes by

$$\delta V = V(x+\delta x, y+\delta y, z+\delta z) - V(x,y,z),$$

and the chain rule tells us that

$$\delta V \simeq \frac{\partial V}{\partial x}\,\delta x + \frac{\partial V}{\partial y}\,\delta y + \frac{\partial V}{\partial z}\,\delta z.$$

Now, the right-hand side of this equation can be rewritten as a scalar product:

$$\delta V \simeq \left(\frac{\partial V}{\partial x}\,\mathbf{i} + \frac{\partial V}{\partial y}\,\mathbf{j} + \frac{\partial V}{\partial z}\,\mathbf{k}\right) \cdot (\delta x\,\mathbf{i} + \delta y\,\mathbf{j} + \delta z\,\mathbf{k}).$$

But

$$\frac{\partial V}{\partial x}\,\mathbf{i} + \frac{\partial V}{\partial y}\,\mathbf{j} + \frac{\partial V}{\partial z}\,\mathbf{k} = \boldsymbol{\nabla} V$$

is the gradient of V, and

$$\delta x\,\mathbf{i} + \delta y\,\mathbf{j} + \delta z\,\mathbf{k} = \delta\mathbf{s}$$

is the displacement vector, so we reach the following conclusion.

The small change in a scalar field V due to a small displacement $\delta\mathbf{s}$ is

$$\delta V \simeq \boldsymbol{\nabla} V \cdot \delta\mathbf{s}. \tag{32}$$

If we know the gradient of a scalar field at a given point, we can use equation (32) to estimate the change in V that occurs when we make a small displacement $\delta\mathbf{s}$ away from that point.

If the tiny displacement vector $\delta\mathbf{s}$ covers a distance δs in the direction of the unit vector $\widehat{\mathbf{n}}$, we can write $\delta\mathbf{s} = \widehat{\mathbf{n}}\,\delta s$, so

$$\delta V \simeq (\boldsymbol{\nabla} V \cdot \widehat{\mathbf{n}})\,\delta s.$$

Then dividing both sides by δs and taking the limit as the distance δs tends to zero, we see that

In the limit where δs tends to zero, our approximations become exact.

$$\frac{dV}{ds} = \boldsymbol{\nabla} V \cdot \widehat{\mathbf{n}}. \tag{33}$$

The rate of change of V with distance in the direction of the unit vector $\widehat{\mathbf{n}}$ is the *component* of the gradient vector $\boldsymbol{\nabla} V$ in the direction of $\widehat{\mathbf{n}}$.

Exercise 14

A scalar field takes the form $V(x,y,z) = (x^2+y^2+z^2)^{3/2}$.

(a) Estimate the change in V between the points $(1,2,2)$ and $(0.98, 1.99, 2.01)$.

(b) Find the rate of change of V at the point $(1,2,2)$ in the direction of the unit vector $\widehat{\mathbf{n}} = (3\,\mathbf{i} + 4\,\mathbf{k})/5$.

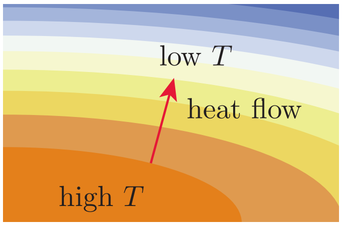


Figure 24 Heat flows in the opposite direction to the gradient of temperature

Gradients of scalar fields in the real world

Gradients often appear in mathematical descriptions of physical systems. You know that heat flows from the hot ring of an electric stove to a colder saucepan, warming its contents. In general, heat flows from regions of high temperature to regions of low temperature, a process that tends to reduce spatial variations in temperature.

For any temperature field $T(x, y, z)$, the temperature gradient $\mathbf{grad}\, T$ plays a vital role in determining how heat is conducted.

As indicated in Figure 24, heat flows in the direction in which temperature decreases most rapidly, which is the direction of $-\mathbf{grad}\, T$. Moreover, the rate of flow of heat at each point is proportional to the magnitude of $\mathbf{grad}\, T$. We can therefore say that

$$\text{heat flow} \propto -\mathbf{grad}\, T.$$

Similar results apply when pressure P or molecular concentration C vary in space. Again, flows arise that tend to reduce these spatial variations, and which are proportional to $-\mathbf{grad}\, P$ and $-\mathbf{grad}\, C$.

Gradients are also important in mechanics. Physicists use a concept called *potential energy*. For example, close to the Earth's surface, the gravitational potential energy of an object of mass m is $V = mgz$, where g is a constant and z is the object's height above the ground. Whenever we are given a potential energy field $V(x, y, z)$, we can deduce the corresponding force by taking minus its gradient. In the case of terrestrial gravity,

$$\mathbf{F} = -\mathbf{grad}\,(mgz) = -mg\,\mathbf{k},$$

which is a constant force that acts vertically downwards (Figure 25).

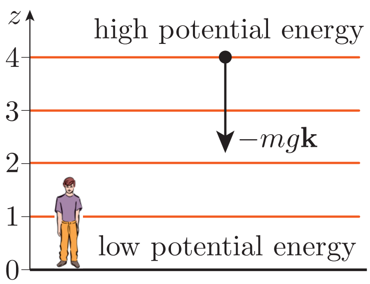


Figure 25 The gravitational force acts downwards, in the opposite direction to the gradient of potential energy

2.2 Gradients in non-Cartesian coordinates

In situations where there is axial or spherical symmetry it is usually best to describe a scalar field in polar, cylindrical or spherical coordinates. In this subsection we explain how to derive the corresponding gradient vector fields, also in polar, cylindrical or spherical coordinates.

We therefore consider a general three-dimensional coordinate system with coordinates (u, v, w) and corresponding unit vectors $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_w$. We assume that the coordinate system is orthogonal, which means that these three unit vectors are mutually orthogonal. If you want to keep a specific example in mind, you can imagine that (u, v, w) stand for (r, θ, ϕ) of spherical coordinates, but the strength of our argument is that (u, v, w) can represent any orthogonal coordinate system.

If we are given a scalar field V, this can be expressed either in Cartesian coordinates or in the (u, v, w) coordinate system. If (x, y, z) and (u, v, w)

label the same point, we can write

$$V(x,y,z) = V(u,v,w).$$

The gradient of V is a vector field, which can be expressed in Cartesian coordinates as

$$\nabla V = \frac{\partial V}{\partial x}\,\mathbf{i} + \frac{\partial V}{\partial y}\,\mathbf{j} + \frac{\partial V}{\partial z}\,\mathbf{k}. \tag{34}$$

The same vector field can also be expressed in the (u,v,w) coordinate system. We do not know the components of ∇V in this system, but we can write

$$\nabla V = g_u\,\mathbf{e}_u + g_v\,\mathbf{e}_v + g_w\,\mathbf{e}_w,$$

where the components g_u, g_v and g_w remain to be determined.

Now, the crucial step is to recall that the gradient of a *scalar field* is a *vector field* – a field whose vector values at each point are independent of the orientation of the coordinate system. This means that we can use Procedure 1 to write, for example,

$$g_u = \mathbf{e}_u \cdot \nabla V.$$

We then evaluate this scalar product using Cartesian expressions for $\mathbf{e}_u$ and ∇V. In Subsection 1.5 you saw that there is a general expression for $\mathbf{e}_u$. According to equation (27),

$$\mathbf{e}_u = \frac{1}{h_u}\left(\frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j} + \frac{\partial z}{\partial u}\,\mathbf{k}\right),$$

where h_u is the scale factor for the u-coordinate. Combining this with the Cartesian expression for ∇V given in equation (34), we get

$$\begin{aligned} g_u &= \frac{1}{h_u}\left(\frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j} + \frac{\partial z}{\partial u}\,\mathbf{k}\right) \cdot \left(\frac{\partial V}{\partial x}\,\mathbf{i} + \frac{\partial V}{\partial y}\,\mathbf{j} + \frac{\partial V}{\partial z}\,\mathbf{k}\right) \\ &= \frac{1}{h_u}\left(\frac{\partial V}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial V}{\partial z}\frac{\partial z}{\partial u}\right). \end{aligned}$$

Finally, the term in brackets can now be simplified using a version of the chain rule (given in Unit 7, equation (24)). We conclude that

$$g_u = \frac{1}{h_u}\frac{\partial V}{\partial u},$$

and there are similar results for the other two components, g_v and g_w. We therefore reach the following very useful conclusion.

Gradient in a general orthogonal coordinate system

Given an orthogonal coordinate system (u,v,w) with unit vectors $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_w$, and scale factors h_u, h_v and h_w, the gradient of a scalar field $V(u,v,w)$ is

$$\nabla V = \frac{1}{h_u}\frac{\partial V}{\partial u}\,\mathbf{e}_u + \frac{1}{h_v}\frac{\partial V}{\partial v}\,\mathbf{e}_v + \frac{1}{h_w}\frac{\partial V}{\partial w}\,\mathbf{e}_w. \tag{35}$$

For a scalar field $V(u,v)$ in two dimensions, the same formula applies, but with the last term (involving $\mathbf{e}_w$) omitted.

This remarkable result applies in all orthogonal coordinate systems. This includes the Cartesian, polar, cylindrical and spherical coordinates studied in this module, as well as other orthogonal coordinate systems that are in occasional use.

In two- and three-dimensional Cartesian coordinate systems, the scale factors are all equal to 1, and equation (35) gives

$$\boldsymbol{\nabla} V = \frac{\partial V}{\partial x}\,\mathbf{e}_x + \frac{\partial V}{\partial y}\,\mathbf{e}_y \quad \text{in two dimensions,}$$

$$\boldsymbol{\nabla} V = \frac{\partial V}{\partial x}\,\mathbf{e}_x + \frac{\partial V}{\partial y}\,\mathbf{e}_y + \frac{\partial V}{\partial z}\,\mathbf{e}_z \quad \text{in three dimensions.}$$

Here, of course, $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are Cartesian unit vectors in the x-, y- and z-directions, more usually written as $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. So we recover the definition of gradient in Cartesian coordinates in equations (30) and (31).

In polar, cylindrical or spherical coordinates, the key to using equation (35) is to know the appropriate scale factors. These were given in Unit 8 in the context of finding area or volume elements for multiple integrals. For ease of reference, the results are summarised below.

Table 1 Scale factors of common coordinate systems

Coordinate system		Scale factors
Polar coordinates	(r,ϕ)	$h_r = 1,\ h_\phi = r$
Cylindrical coordinates	(r,ϕ,z)	$h_r = 1,\ h_\phi = r,\ h_z = 1$
Spherical coordinates	(r,θ,ϕ)	$h_r = 1,\ h_\theta = r,\ h_\phi = r\sin\theta$

Using Table 1 and equation (35), we can immediately write down the expression for gradient in polar coordinates. With $h_r = 1$ and $h_\phi = r$, we get

$$\boldsymbol{\nabla} V = \frac{\partial V}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\,\mathbf{e}_\phi. \tag{36}$$

You can use this formula to find the gradient of any scalar field given in polar coordinates.

Example 5

The domains of V and $\boldsymbol{\nabla} V$ exclude the origin $r = 0$.

Exercise 11 considered the scalar field $V(x,y) = \ln(\sqrt{x^2+y^2})$ in Cartesian coordinates. In polar coordinates, this field takes the form $V(r,\phi) = \ln(r)$. Find the gradient of $V(r,\phi)$ in polar coordinates.

Solution

The partial derivatives of $V(r,\phi) = \ln(r)$ are

$$\frac{\partial V}{\partial r} = \frac{1}{r} \quad \text{and} \quad \frac{\partial V}{\partial \phi} = 0,$$

so for this scalar field,

$$\nabla V = \frac{1}{r}\,\mathbf{e}_r + \frac{1}{r}\,0\,\mathbf{e}_\phi = \frac{1}{r}\,\mathbf{e}_r \quad (r \neq 0).$$

This gradient field has magnitude $1/r$ and points radially outwards, away from the origin. This answer agrees with that of Exercise 11, but has been obtained more efficiently.

Using Table 1 and equation (35), we can also find expressions for gradient in cylindrical coordinates and spherical coordinates.

In *cylindrical coordinates*, the scale factors are $h_r = 1$, $h_\phi = r$ and $h_z = 1$, so

$$\nabla V = \frac{\partial V}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\,\mathbf{e}_\phi + \frac{\partial V}{\partial z}\,\mathbf{e}_z. \tag{37}$$

In *spherical coordinates*, the scale factors are $h_r = 1$, $h_\theta = r$ and $h_\phi = r\sin\theta$, so

$$\nabla V = \frac{\partial V}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\,\mathbf{e}_\phi. \tag{38}$$

You should not bother to memorise these results, as it is easier to recall the general shape of equation (35) and the relevant scale factors.

Exercise 15

Exercise 12 considered the scalar field $V(x,y,z) = 1/(x^2+y^2+z^2)^{3/2}$ in Cartesian coordinates. In spherical coordinates, this field takes the form

$$V(r,\theta,\phi) = \frac{1}{r^3}.$$

Calculate the gradient of $V(r,\theta,\phi)$ in spherical coordinates.

The domains of V and ∇V exclude the origin $r = 0$.

Exercise 16

In cylindrical coordinates, a scalar field takes the form

$$f(r,\phi,z) = r^2\sin(2\phi) + z^2.$$

Calculate the gradient ∇f, and hence find the magnitude of the gradient at any point.

Exercise 17

In spherical coordinates, a scalar field takes the form

$$T(r,\theta,\phi) = r\sin\theta.$$

(a) Find the corresponding gradient vector field.

(b) Find the rate of change of T with distance in the direction of the unit vector $\widehat{\mathbf{n}} = (\mathbf{e}_\theta + \mathbf{e}_\phi)/\sqrt{2}$.

2.3 The del operator

The process that leads from a scalar field to a gradient field can be described using a concept called the *del operator*. For the moment, this just repackages what you already know, but you will soon see that the del operator is useful in other contexts.

Essentially, an *operator* is something that 'acts on things to produce other things'. For example, a particular rotation operator may represent a rotation by 30° about the z-axis; when this operator acts on any position vector, it produces another position vector.

We are interested in operators that act on functions to produce other functions. An example is the differentiation operator d/dx. This acts on any differentiable function $f(x)$ to produce another function, $f'(x)$.

In Cartesian coordinates, the **del operator** is defined by

$$\boldsymbol{\nabla} = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}. \tag{39}$$

When this operator acts on a scalar field $V(x, y, x)$ expressed in Cartesian coordinates, it produces the gradient vector field

$$\boldsymbol{\nabla} V = \frac{\partial V}{\partial x}\,\mathbf{i} + \frac{\partial V}{\partial y}\,\mathbf{j} + \frac{\partial V}{\partial z}\,\mathbf{k},$$

in agreement with equation (30).

When the del operator acts on a given scalar field, it produces a definite gradient vector field. However, the scalar field and the resulting gradient vector field can be described in various coordinate systems. The form chosen to represent the del operator also depends on the coordinate system used. In an orthogonal coordinate system (u, v, w) with unit vectors $\mathbf{e}_u$, $\mathbf{e}_v$, $\mathbf{e}_w$ and scale factors h_u, h_v, h_w, the del operator is given by

$$\boldsymbol{\nabla} = \mathbf{e}_u \frac{1}{h_u}\frac{\partial}{\partial u} + \mathbf{e}_v \frac{1}{h_v}\frac{\partial}{\partial v} + \mathbf{e}_w \frac{1}{h_w}\frac{\partial}{\partial w}. \tag{40}$$

When this operator acts on a scalar field $V(u, v, w)$ expressed in (u, v, w) coordinates, it produces the gradient vector field

$$\boldsymbol{\nabla} V = \mathbf{e}_u \frac{1}{h_u}\frac{\partial V}{\partial u} + \mathbf{e}_v \frac{1}{h_v}\frac{\partial V}{\partial v} + \mathbf{e}_w \frac{1}{h_w}\frac{\partial V}{\partial w},$$

in agreement with equation (35).

Notice that the unit vectors in equation (40) have been placed to the left of the partial derivative operators $\partial/\partial u$, $\partial/\partial v$ and $\partial/\partial w$. This is a necessary precaution because the unit vectors generally depend on the coordinates (u, v, w). If they were placed to the right of the partial derivative operators, we would need to differentiate them, and this would not give the correct gradient vector field. Cartesian coordinates are a special case because the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are all constant vectors, so they can be placed either to the left or to the right of $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$.

Because the del operator contains unit vectors, it should be thought of as having a vectorial (rather than scalar) character. It is sometimes described as being a *vector differential operator*. That is why it is printed in bold. In written work, you should underline it with a wavy or straight line.

Exercise 18

Use equation (40) and the scale factors of Table 1 to express the del operator in polar coordinates, cylindrical coordinates and spherical coordinates.

3 The divergence of a vector field

For a scalar field $V(x, y, z)$, the three partial derivatives

$$\frac{\partial V}{\partial x}, \quad \frac{\partial V}{\partial y}, \quad \frac{\partial V}{\partial z}$$

describe its spatial rates of change in the x-, y- and z-directions. However, you have seen that it is useful to group these three partial derivatives together to form the gradient field $\boldsymbol{\nabla} V$. Rather than thinking about the separate partial derivatives, we can think about the gradient field, which has a magnitude and direction at each point. This is a powerful idea – you have seen that the gradient field allows us to calculate the spatial rate of change of V in *any* direction (not just the coordinate directions).

The rest of this unit discusses the spatial rates of change of *vector fields*. A three-dimensional vector field $\mathbf{F}(x, y, z)$ has three components, F_x, F_y and F_z, so there are nine partial derivatives to consider at each point:

$$\frac{\partial F_x}{\partial x}, \frac{\partial F_x}{\partial y}, \frac{\partial F_x}{\partial z}, \quad \frac{\partial F_y}{\partial x}, \frac{\partial F_y}{\partial y}, \frac{\partial F_y}{\partial z}, \quad \frac{\partial F_z}{\partial x}, \frac{\partial F_z}{\partial y}, \frac{\partial F_z}{\partial z}.$$

This is a great deal of information, making it hard to visualise what is going on. Fortunately, the nine partial derivatives can be grouped into two significant quantities – called the *divergence* and the *curl* of the vector field. In most cases, these two quantities tell us all we need to know about the spatial rates of change of a vector field.

The fact that the del operator $\boldsymbol{\nabla}$ has a vectorial character suggests two ways of grouping the partial derivatives. Given two vectors $\mathbf{a}$ and $\mathbf{b}$, we can define two different types of product. The scalar product is

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z,$$

and the vector product is

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\,\mathbf{i} + (a_z b_x - a_x b_z)\,\mathbf{j} + (a_x b_y - a_y b_x)\,\mathbf{k}.$$

For the del operator $\boldsymbol{\nabla}$ acting on a vector field $\mathbf{F}(x,y,z)$, we can introduce the corresponding combinations

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

and

$$\boldsymbol{\nabla}\times\mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}.$$

It turns out that these are precisely the combinations that we need: $\boldsymbol{\nabla}\cdot\mathbf{F}$ is called the **divergence** of $\mathbf{F}$, and $\boldsymbol{\nabla}\times\mathbf{F}$ is called the **curl** of $\mathbf{F}$. This section discusses divergence, while Section 4 discusses curl.

3.1 Divergence in Cartesian coordinates

The above discussion gave a broad overview. Here we begin afresh with the basic definition of divergence in Cartesian coordinates.

Divergence of a vector field in Cartesian coordinates

Suppose that a vector field $\mathbf{F}$ is expressed in Cartesian coordinates as

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}.$$

Then the divergence of $\mathbf{F}$ is defined as

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}. \tag{41}$$

The alternative notation $\operatorname{div}\mathbf{F}$ is sometimes used for divergence, so we can also write

$$\operatorname{div}\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}. \tag{42}$$

Two questions naturally arise: how is divergence calculated, and what does divergence tell us? We begin with the calculations, and then discuss the interpretation.

Calculating divergence

It is easy to calculate the divergence of a vector field $\mathbf{F}$ in Cartesian coordinates. All you need to do is to identify the Cartesian components F_x, F_y and F_z of the field, find their partial derivatives with respect to the corresponding coordinates x, y and z, and add the results together.

We begin with two-dimensional vector fields because they are simpler to visualise.

Example 6

Find the divergence of each of the following vector fields:

$$\mathbf{A} = 3\,\mathbf{i} + 2\,\mathbf{j},\quad \mathbf{B} = y\,\mathbf{i} - x\,\mathbf{j},\quad \mathbf{C} = x\,\mathbf{i} + y\,\mathbf{j},\quad \mathbf{D} = x^2\,\mathbf{i} - y^2\,\mathbf{j}.$$

Solution

Using the definition of divergence (equation (41)), we get

$$\begin{aligned}
\nabla \cdot \mathbf{A} &= \frac{\partial(3)}{\partial x} + \frac{\partial(2)}{\partial y} + \frac{\partial(0)}{\partial z} = 0, \\
\nabla \cdot \mathbf{B} &= \frac{\partial(y)}{\partial x} + \frac{\partial(-x)}{\partial y} + \frac{\partial(0)}{\partial z} = 0, \\
\nabla \cdot \mathbf{C} &= \frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(0)}{\partial z} = 2, \\
\nabla \cdot \mathbf{D} &= \frac{\partial(x^2)}{\partial x} + \frac{\partial(-y^2)}{\partial y} + \frac{\partial(0)}{\partial z} = 2(x - y).
\end{aligned}$$

This example illustrates some cases that can arise. For vector fields **A** and **B**, the divergence is equal to zero everywhere. The vector field **C** has a constant non-zero divergence, and **D** has a divergence that varies with position, which is the most usual situation. The divergence is always a scalar function of position (which may be a constant or zero).

It is important to understand the distinction between gradient and divergence. Given a scalar field V, we can construct its gradient ∇V, which has vector values:

$$\nabla V = \frac{\partial V}{\partial x}\,\mathbf{i} + \frac{\partial V}{\partial y}\,\mathbf{j} + \frac{\partial V}{\partial z}\,\mathbf{k}.$$

Given a vector field **F**, we can construct its divergence $\nabla \cdot \mathbf{F}$, which has scalar values:

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

The gradient is a vector, so its expression involves unit vectors. By contrast, divergence is a scalar, and its expression is just a sum of derivatives with no unit vectors. When you specify $\nabla \cdot \mathbf{F}$ in handwriting, you must underline both ∇ and **F** *and* include a dot between them – otherwise your reader may think that you are referring to the gradient of a scalar field F.

Exercise 19

Calculate the divergence of each of the following vector fields.

(a) $\mathbf{F}(x, y, z) = x^2 y\,\mathbf{i} + y^2 z\,\mathbf{j} - yz^2\,\mathbf{k}$

(b) $\mathbf{G}(x, y, z) = (x + y)^2\,\mathbf{i} + (y + z)^2\,\mathbf{j} + (x + z)^2\,\mathbf{k}$

Exercise 20

A scalar field V takes the form $V = x^4 + y^4 + z^4$.

(a) Would it make sense to take the divergence of this field?

(b) Find the gradient field ∇V, and calculate its divergence, $\nabla \cdot (\nabla V)$.

Interpreting divergence

At any given point (x, y, z), the divergence of a vector field has a definite scalar value. In fact, we can go further.

It turns out that the divergence of a vector field is a *scalar field*.

This is a important fact. Recalling the definition of a scalar field given in Subsection 1.1, it means that the value of the divergence at any given point is *independent of the orientation of the coordinate system*. This suggests that the divergence of a vector field might describe some significant property of the vector field. Indeed it does, and the name *divergence* provides a clue.

Intuitive meaning of divergence

The divergence of a vector field $\mathbf{F}$ describes the extent to which $\mathbf{F}$ flows outwards or diverges from each point.

This statement is not precise. Deciding how to quantify the 'extent of outward flow' and linking this to the definition of divergence in equation (41) involves a lengthy discussion, and the details are left for the next unit. For the moment, we just consider a few typical examples. The aim is to give you an intuitive feeling for divergence so that you can interpret the results of your calculations.

In fact, $\mathbf{J}$ is the fluid density times the fluid velocity at each point. In this context, $\mathbf{J}$ is not the Jacobian vector of Unit 8.

Consider a vector field $\mathbf{J}(x, y, z)$ that describes the flow of mass in a fluid. The fluid could be water or air, and it may have a fixed or variable density. The precise definition of $\mathbf{J}$ is not needed in this informal discussion.

The cube should be tiny: strictly speaking, we are interested in the limiting case where its size tends to zero.

At a point P, the value of $\boldsymbol{\nabla} \cdot \mathbf{J}$ tells us about the net flow of fluid towards or away from P. If we draw a tiny cube around P, then the divergence of $\mathbf{J}$ at P is:

- positive if more fluid leaves the cube than enters it
- negative if more fluid enters the cube than leaves it
- equal to zero if the outflow exactly matches the inflow.

Similar remarks apply to two-dimensional velocity fields, but with the cube replaced by a square.

These ideas can be illustrated with the vector fields of Example 6. Arrow maps for the fields $\mathbf{A} = 3\,\mathbf{i} + 2\,\mathbf{j}$ and $\mathbf{B} = y\,\mathbf{i} - x\,\mathbf{j}$ are shown in Figure 26, with selected points indicated by blue dots and small blue squares drawn around them. Example 6 showed that these fields have zero divergence everywhere. So according to our interpretation of divergence, there should be no net flow into or out of these squares. So far as it is possible to tell, the arrow maps confirm this interpretation.

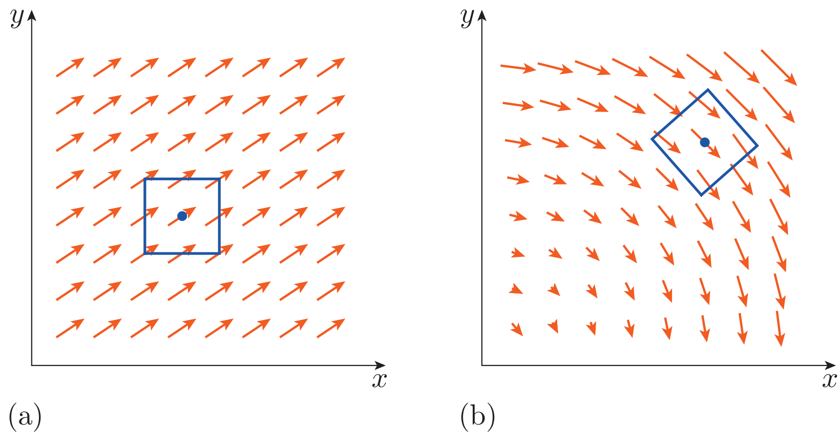


Figure 26 Arrow maps for the vector fields $\mathbf{A}$ and $\mathbf{B}$ of Example 6

Arrow maps for the vector fields $\mathbf{C} = x\,\mathbf{i} + y\,\mathbf{j}$ and $\mathbf{D} = x^2\,\mathbf{i} - y^2\,\mathbf{j}$ of Example 6 are shown in Figure 27. Example 6 showed that $\mathbf{C}$ has a divergence that is equal to 2 everywhere. This positive divergence corresponds to the fact that there is a net flow out of the blue square in Figure 27(a). The vector field $\mathbf{D}$ has a divergence equal to $2(x - y)$, which is positive for $x > y$ and negative for $x < y$. Our interpretation of divergence is supported by Figure 27(b), which shows a net flow out of the lower square, and a net flow into the upper square.

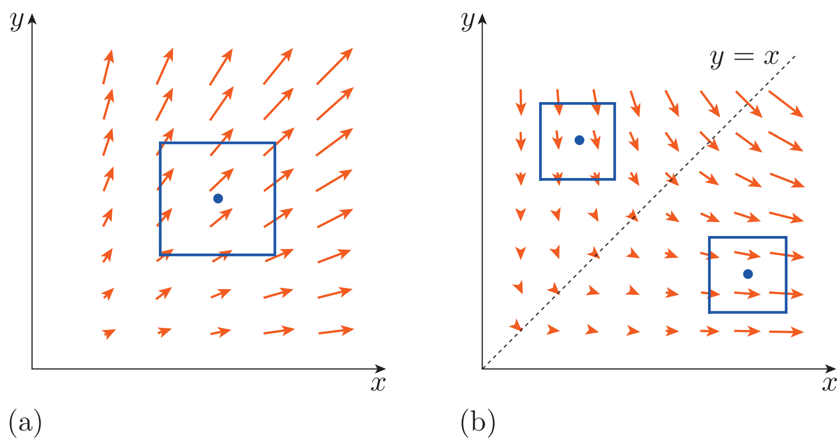


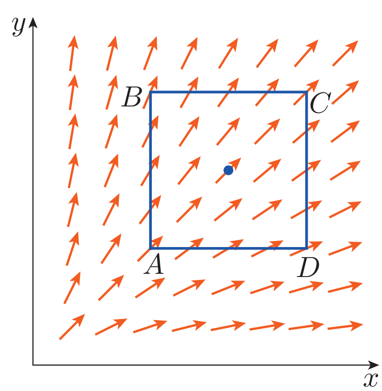
Figure 27 Arrow maps for the vector fields $\mathbf{C}$ and $\mathbf{D}$ of Example 6

Exercise 21

(a) Calculate the divergence of the vector field

$$\mathbf{F} = \frac{x}{\sqrt{x^2 + y^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\,\mathbf{j} \quad ((x, y) \neq (0, 0)).$$

(b) The arrow map for $\mathbf{F}$ is shown in the margin. Use the square $ABCD$ to show that this diagram supports our interpretation of divergence.



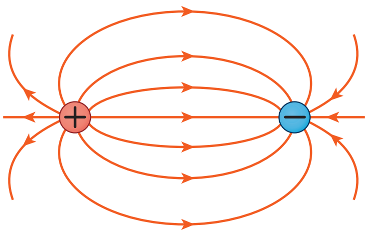


Figure 28 The electric field around positive and negative charge distributions

Divergences of vector fields in the real world

Not all vector fields describe flows. A *gravitational field*, for example, quantifies the gravitational influence of massive objects. The divergence of a gravitational field turns out to be negative at points occupied by matter (inside the Earth, for example, or inside the Sun). In empty space, the divergence of a gravitational field is equal to zero.

A similar situation applies to *electric fields* (Figure 28). The divergence of an electric field is positive at points where there is positive charge, and negative at points where there is negative charge. In empty space, the divergence of an electric field is equal to zero.

Magnetic fields are unusual: so far as we know, the divergence of any magnetic field is equal to zero everywhere.

3.2 Divergence in non-Cartesian coordinates

You have seen that some vector fields are best described in non-Cartesian coordinate systems. For example, the gravitational field around the Sun points radially inwards towards the Sun, and is most naturally described in spherical coordinates. We often have to calculate the divergence of such vector fields, so we need to know how to calculate divergence in non-Cartesian coordinates. Polar, cylindrical and spherical coordinates are all important in applications. All of these are orthogonal coordinate systems.

In an orthogonal system of coordinates (u, v, w) with unit vectors $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_w$, a vector field $\mathbf{F}$ can be expressed as

$$\mathbf{F} = F_u\,\mathbf{e}_u + F_v\,\mathbf{e}_v + F_w\,\mathbf{e}_w,$$

and the del operator is given by

$$\boldsymbol{\nabla} = \mathbf{e}_u\,\frac{1}{h_u}\,\frac{\partial}{\partial u} + \mathbf{e}_v\,\frac{1}{h_v}\,\frac{\partial}{\partial v} + \mathbf{e}_w\,\frac{1}{h_w}\,\frac{\partial}{\partial w}.$$

It follows that the divergence of $\mathbf{F}$ in the (u, v, w) coordinate system is

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \left(\mathbf{e}_u\,\frac{1}{h_u}\,\frac{\partial}{\partial u} + \mathbf{e}_v\,\frac{1}{h_v}\,\frac{\partial}{\partial v} + \mathbf{e}_w\,\frac{1}{h_w}\,\frac{\partial}{\partial w}\right)\cdot\left(F_u\,\mathbf{e}_u + F_v\,\mathbf{e}_v + F_w\,\mathbf{e}_w\right).$$

This formula is correct, but is not immediately useful because there are still partial derivatives and scalar products to evaluate. The situation is complicated by the fact that unit vectors such as $\mathbf{e}_u$ may depend on the coordinates (u, v, w). So when the partial differentials act on the right-hand bracket, the unit vectors $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_w$ must be differentiated as well as the components F_u, F_v and F_w. The calculations get very messy!

In practice, scientists and mathematicians do not spend time deriving general formulas for divergence in non-Cartesian coordinates. Life is too short, so they simply look up the results they need in reference works.

We take a similar attitude. The Appendix to this unit justifies the expressions that we will use, but this is optional material, and will not be assessed or examined. We focus here on the more important (and more straightforward) task of stating and applying standard formulas.

Equation (35) gave a general formula for the gradient of a scalar field in any orthogonal coordinate system. Rather wonderfully, in spite of the complications noted above, there is a corresponding formula for the divergence of a vector field in any orthogonal coordinate system.

Divergence of a vector field in orthogonal coordinates

In any orthogonal coordinate system (u, v, w) with scale factors h_u, h_v and h_w, a vector field $\mathbf{F}$ has divergence

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{J}\left[\frac{\partial}{\partial u}\left(\frac{JF_u}{h_u}\right) + \frac{\partial}{\partial v}\left(\frac{JF_v}{h_v}\right) + \frac{\partial}{\partial w}\left(\frac{JF_w}{h_w}\right)\right], \qquad (43)$$

where

$$J = h_u h_v h_w$$

is the product of the scale factors, called the **Jacobian factor**.

In a two-dimensional orthogonal system (u, v), the same formula applies, but with the last term omitted, and $J = h_u h_v$.

J also appeared in Unit 8 in the context of volume integrals.

It is easy to check that this formula gives the correct result in Cartesian coordinates $(u, v, w) = (x, y, z)$. In this case, all the scale factors are equal to 1, so $J = 1$ and equation (43) reduces to

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z},$$

as expected.

In polar coordinates $(u, v) = (r, \phi)$, the scale factors are $h_r = 1$ and $h_\phi = r$, so $J = r$. In this case, equation (43) gives the following formula.

These scale factors are given in Table 1.

Divergence in polar coordinates

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\phi}{\partial \phi}. \qquad (44)$$

When there is rotational symmetry, it is often best to specify two-dimensional vector fields in polar coordinates, and to calculate their divergences using equation (44).

Example 7

Find the divergences of the following two-dimensional vector fields, expressed in polar coordinates.

(a) $\mathbf{F}(r, \phi) = \mathbf{e}_r \ (r \neq 0)$

(b) $\mathbf{G}(r, \phi) = r\,\mathbf{e}_r + r\sin\phi\,\mathbf{e}_\phi$

Solution

(a) The polar components of $\mathbf{F}$ are $F_r = 1$ and $F_\phi = 0$, so

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial(r)}{\partial r} = \frac{1}{r} \quad (r \neq 0).$$

This is the same field as that in Exercise 21. The calculation is much easier in polar, rather than Cartesian coordinates!

(b) The polar components of $\mathbf{G}$ are $G_r = r$ and $G_\phi = r\sin\phi$, so

$$\nabla \cdot \mathbf{G} = \frac{1}{r}\frac{\partial(r^2)}{\partial r} + \frac{1}{r}\frac{\partial(r\sin\phi)}{\partial\phi} = 2 + \cos\phi.$$

Similar methods apply in three dimensions. In *cylindrical coordinates* $(u, v, w) = (r, \phi, z)$, the scale factors are $h_r = 1$, $h_\phi = r$ and $h_z = 1$, so $J = r$. In this case, equation (43) gives

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\left(\frac{\partial(rF_r)}{\partial r} + \frac{\partial F_\phi}{\partial\phi} + \frac{\partial(rF_z)}{\partial z}\right).$$

Since r is treated as a constant when partially differentiating with respect to z, we have the following result.

Divergence in cylindrical coordinates

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}. \tag{45}$$

Not surprisingly, this is similar to the expression for divergence in polar coordinates, but with an additional term, $\partial F_z/\partial z$.

In *spherical coordinates*, the scale factors are $h_r = 1$, $h_\theta = r$ and $h_\phi = r\sin\theta$, so $J = r^2\sin\theta$. Hence

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2\sin\theta}\left(\frac{\partial(r^2\sin\theta\,F_r)}{\partial r} + \frac{\partial(r\sin\theta\,F_\theta)}{\partial\theta} + \frac{\partial(rF_\phi)}{\partial\phi}\right).$$

Remembering that functions of one variable are treated as constants when partially differentiating with respect to another variable, we get the following formula.

Divergence in spherical coordinates

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,F_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}. \tag{46}$$

You can take all these results on trust. Equations (44)–(46) are all listed in the Handbook, so you need not memorise them for the exam. For more general purposes, it is worth trying to remember equation (43), which has a more symmetrical shape than the others, and can be used to construct them all.

The focus here is on *using* equations (44)–(46) to find the divergences of given vector fields. You will generally be told which coordinate system is used, so you just need to select the appropriate formula for divergence, carry out the partial differentiations, and simplify the result if possible.

Exercise 22

The following fields are in cylindrical coordinates. Find their divergences.

(a) $\mathbf{F}(r,\phi,z) = 4r^3\,\mathbf{e}_r$ (b) $\mathbf{G}(r,\phi,z) = r^2\sin\phi\,\mathbf{e}_r + z^2\,\mathbf{e}_z$

It is essential to state which coordinate system is used because the coordinate r has different meanings in the cylindrical and spherical systems.

Exercise 23

The following fields are in spherical coordinates. Find their divergences.

(a) $\mathbf{F}(r,\theta,\phi) = 4r^3\,\mathbf{e}_r$ (b) $\mathbf{G}(r,\theta,\phi) = r\sin^2\theta\,\mathbf{e}_\theta + r\cos\theta\cos\phi\,\mathbf{e}_\phi$

Exercise 24

In spherical coordinates, a vector field $\mathbf{F}$ takes the form $\mathbf{F} = f(r)\,\mathbf{e}_r$, where $f(r)$ depends only on r, the distance from the origin. If $\operatorname{div}\mathbf{F} = 0$ at all points except the origin, show that $f(r)$ is proportional to $1/r^2$.

4 The curl of a vector field

The second important quantity describing the spatial rate of change of a vector field $\mathbf{F}$ is its curl. In this section, we introduce curl in Cartesian coordinates and illustrate its physical meaning with some typical vector fields. We also show how curl is calculated in non-Cartesian orthogonal coordinate systems.

4.1 Curl in Cartesian coordinates

We briefly mentioned curl when considering ways in which the operator $\boldsymbol{\nabla}$ can act on vector fields. Here we begin afresh with the basic definition.

Curl of a vector field in Cartesian coordinates

Suppose that a vector field $\mathbf{F}$ is expressed in Cartesian coordinates as

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}.$$

Then the curl of $\mathbf{F}$ is defined as

$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}. \qquad (47)$$

The alternative notation $\mathbf{curl}\,\mathbf{F}$ is commonly used.

As always when dealing with vector products, we assume that the coordinate system is right-handed.

In this definition, the partial derivative operators in the second row act on the components in the third row. The determinant can then be expanded in the usual way. For example, the term involving $\mathbf{i}$ is

$$\mathbf{i}\left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right) = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i}.$$

The complete expansion gives the vector quantity

$$\boldsymbol{\nabla} \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}. \quad (48)$$

It is worth comparing the definitions of divergence and curl in equations (41) and (48). The derivatives that occur in divergence may be said to 'go with the components' – for example, $\partial/\partial x$ acts on F_x, and so on. Curl is quite different: it is built up of partial derivatives such as $\partial F_x/\partial y$ and $\partial F_y/\partial x$ that describe how rapidly a component in one direction changes when we move in a *perpendicular* direction. In three dimensions, there are six such derivatives, and these are arranged in pairs to give the components of $\boldsymbol{\nabla} \times \mathbf{F}$ shown in equation (48).

We will calculate curls shortly. This is a straightforward task – we just need to calculate and combine the appropriate partial derivatives. Before doing this, let us see what curl means. Clearly, at any given point, $\boldsymbol{\nabla} \times \mathbf{F}$ has a definite value. In fact, we can go further.

By contrast, the divergence of a vector field is a scalar field.

It turns out that the curl of a vector field is another *vector field.*

From the definition of a vector field (see Subsection 1.1), this implies that at any given point, the magnitude and direction of $\boldsymbol{\nabla} \times \mathbf{F}$ are *independent of the orientation of the coordinate system.* This suggests that $\boldsymbol{\nabla} \times \mathbf{F}$ might have some significant physical meaning. This is indeed the case, and the name *curl* provides a good clue.

Intuitive meaning of curl

The curl of a vector field $\mathbf{F}$ describes the extent to which $\mathbf{F}$ rotates or swirls locally about each point.

In three-dimensional space, rotation involves an axis of rotation and a sense of rotation about that axis. Taking the coordinate system to be right-handed, these features are related to curl in a simple and direct way.

- The axis of local rotation is along the direction of the curl vector.
- The sense of rotation around this axis is found by the **right-hand grip rule** illustrated in Figure 29. With the thumb of your right hand pointing in the direction of $\boldsymbol{\nabla} \times \mathbf{F}$, the fingers of your closed right hand indicate the sense of rotation associated with $\mathbf{F}$.

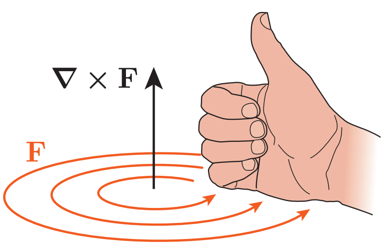


Figure 29 The right-hand grip rule

Establishing a precise link between the concept of 'local rotation' and the definition of curl is left for the next unit. For the moment, we just consider a few typical examples. The aim is to give you an intuitive feeling for curl so that you can interpret the results of your calculations.

First, we consider two-dimensional vector fields. A vector field in the xy-plane takes the form

$$\mathbf{V} = V_x(x, y)\,\mathbf{i} + V_y(x, y)\,\mathbf{j}.$$

In this case, $V_z = 0$, $\partial V_x/\partial z = 0$ and $\partial V_y/\partial z = 0$. Substituting into equation (48), we see that the curl of $\mathbf{V}$ has only one component:

$$\boldsymbol{\nabla} \times \mathbf{V} = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\mathbf{k}. \qquad (49)$$

One way of interpreting such a curl is to suppose that $\mathbf{V}$ describes the velocity of water on the surface of a river. Imagine a tiny circular disc floating on the water. The disc will be carried downstream following the direction of flow, but it may also rotate about a vertical axis as it drifts. The curl of $\mathbf{V}$ is proportional to the rate of rotation of the disc. Of course, the disc just serves as a marker making the curl of the underlying vector field visible – we are not really interested in the disc itself.

To take a specific example, Figure 30 shows a straight stretch of river with its banks at $y = -4$ and $y = 4$ (in metres), with water flowing in the x-direction. Suppose that the velocity of water on the surface of this river (in metres per second) is given by the vector field

$$\mathbf{V} = (16 - y^2)\,\mathbf{i} \quad (-4 \le y \le 4). \qquad (50)$$

Then the curl of this vector field is

$$\mathbf{curl}\,\mathbf{V} = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\mathbf{k} = 2y\,\mathbf{k}, \qquad (51)$$

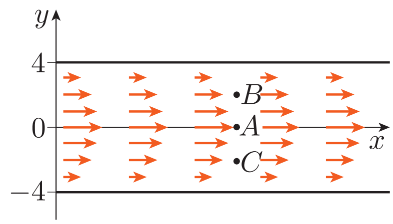


Figure 30 The flow of water in a river

which varies from point to point. Remember that our conventions require us to use right-handed coordinate systems, so with the x- and y-axes as shown in Figure 30, the z-axis points out of the page, towards you. Bearing this in mind, the interpretation of curl can be checked as follows.

Suppose that a disc is placed at point A in Figure 30, equidistant from either bank. At this point $y = 0$, and equation (51) gives zero curl. This makes good sense because water flows symmetrically around the disc, producing no tendency to rotate one way or another: the disc drifts downstream without rotating.

At point B, $y = 2$ and the calculated curl points along the positive z-axis. Using the right-hand grip rule, this is associated with a rotation about a vertical axis in an anticlockwise sense when viewed from above the river. This correctly describes how the disc revolves in response to a current that is stronger at the centre of the river than near its banks. At point C, $y = -2$ and this conclusion is reversed. The curl now points along the negative z-axis, corresponding to a clockwise rotation of the disc when seen from above, which is again what we should expect.

Even a straight-line flow may be associated with rotation and curl.

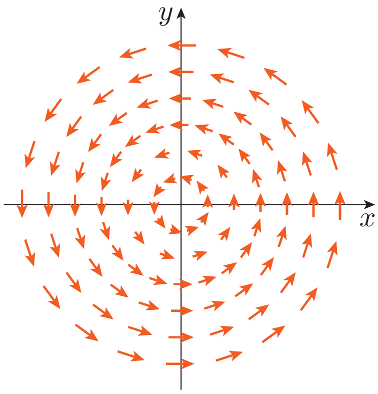


Figure 31 A swirling flow

A good example of a swirling flow is given by the vector field

$$\mathbf{V} = -y\,\mathbf{i} + x\,\mathbf{j},$$

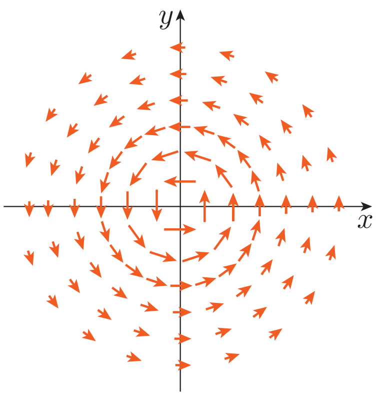
whose arrow map is shown in Figure 31. We have

$$\mathbf{curl}\,\mathbf{V} = \left(\frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y}\right)\mathbf{k} = 2\,\mathbf{k},$$

which is in the positive z-direction (i.e. out of the page towards you).

The result can be interpreted using the right-hand grip rule. With the outstretched thumb of your right hand pointing in the z-direction, your fingers wrap in an anticlockwise sense. This is as expected: if Figure 31 represents a flow of water, then a float placed in the centre of this flow would certainly revolve anticlockwise. In fact, **curl V** is a constant in this case, so the float would revolve anticlockwise, at the same rate, no matter where it was placed in the flow. This happens because water flows unsymmetrically around the float, producing an effect similar to that already described for the straight-flowing river of Figure 30.

It may be tempting to suppose that any vector field with field lines that are closed loops has a non-zero curl. This is not true, as the following exercise shows.



Exercise 25

The vector field

$$\mathbf{V} = -\frac{y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j} \quad ((x,y) \neq (0,0))$$

has the arrow map sketched in the margin. Show that the curl of $\mathbf{V}$ is equal to the zero vector at all points in the domain of $\mathbf{V}$.

The result of this exercise does not contradict our statement that curl describes the local rotation of a vector field. The important word here is *local*. If the field in Exercise 25 describes the two-dimensional flow of water, and a tiny disc is placed on the surface of the water, then the disc will travel around the origin in circles, following the circular field lines. However, for this particular flow, the disc does not rotate *locally*. If the disc is marked with an arrow that initially points East, the arrow continues to point East as the disc drifts in the current, as shown in Figure 32. This absence of *local* rotation agrees with the calculation of zero curl.

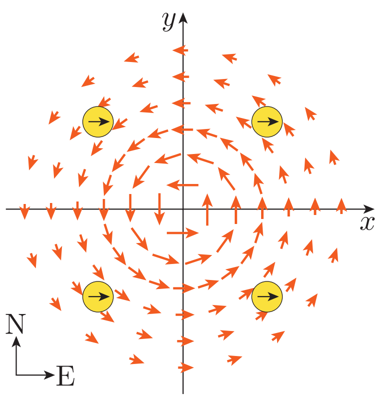


Figure 32 An arrow on a very small disc points in a fixed direction as the disc drifts in the flow of Exercise 25

Curl in three dimensions

So far we have considered the curls of two-dimensional vector fields. More usually, we need to find the curl of a three-dimensional vector field. The interpretation is essentially the same. If a fluid has a velocity field $\mathbf{V}$, then a sphere immersed in the fluid revolves about an axis aligned with the curl of $\mathbf{V}$, and the sense of rotation is given by the right-hand grip rule.

To calculate the curl of any three-dimensional vector field $\mathbf{F}$, we start from the definition:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$

It is a good idea to write down this equation at the start of any calculation of curl in three dimensions. The alternative is to write down the expanded version of the determinant,

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k},$$

but this equation is harder to remember. If you want to use it as your starting point, it is helpful to note that it contains a strong pattern, based on the cyclic ordering shown in Figure 33.

- The x-component is obtained by acting with $\partial/\partial y$ on F_z (note the order $x \to y \to z$); a term with y and z interchanged is then subtracted.
- The y-component is obtained by acting with $\partial/\partial z$ on F_x (note the order $y \to z \to x$); a term with z and x interchanged is then subtracted.
- The z-component is obtained by acting with $\partial/\partial x$ on F_y (note the order $z \to x \to y$); a term with x and y interchanged is then subtracted.

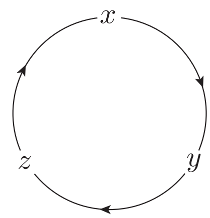


Figure 33 A cyclic ordering of x, y and z underlies the formula for curl in Cartesian coordinates

Exercise 26

Find the curls of the following vector fields.

(a) $\mathbf{F} = (z-y)\,\mathbf{i} + (x-z)\,\mathbf{j} + (y-x)\,\mathbf{k}$ (b) $\mathbf{G} = zy^2\,\mathbf{i} + xz^2\,\mathbf{j} + yx^2\,\mathbf{k}$

Exercise 27

Given a scalar field $U(x, y, z)$, the corresponding gradient vector field is

$$\nabla U = \frac{\partial U}{\partial x}\,\mathbf{i} + \frac{\partial U}{\partial y}\,\mathbf{j} + \frac{\partial U}{\partial z}\,\mathbf{k}.$$

Show that the curl of this gradient vector field vanishes everywhere.

You may assume that U varies smoothly enough to obey the mixed partial derivative theorem of Unit 7.

Curls of vector fields in the real world

Many fluid flows have a swirling motion, known as a vortex. Very often, the curl of the velocity is small over large regions, and is significant only in a small central region called the *vortex core*.

An aircraft in flight continually generates a vortex from each of its wing tips, as illustrated in Figure 34. This is an inevitable consequence of the air flows that generate lift, but introduces unwelcome drag. To improve fuel economy, wings are designed to minimise the energy that is wasted by generating such vortices.

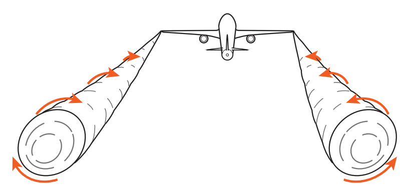


Figure 34 Vortices shed from an aeroplane's wing tips

Figure 35 Migrating geese maintain a V-formation

You may have seen birds such as geese flying in a V-formation (Figure 35). This is also an adaptation to the generation of vortices at wing tips, because the lagging birds benefit from the upward-flowing air in vortices generated by the leading bird. Fair play is observed, as the birds regularly switch positions in the formation.

Electric and magnetic fields can also have curl, and this is vital for many phenomena. A magnetic field that changes with time produces an electric field with a non-zero curl, and this lies behind the functioning of electricity generators. Moreover, an electric field that changes with time produces a magnetic field with a non-zero curl. These facts underpin the interpretation of light as a travelling disturbance of electric and magnetic fields.

4.2 Curl in non-Cartesian coordinates

You have seen that some vector fields are best described in non-Cartesian coordinate systems. This subsection explains how to calculate the curl of a vector field expressed in any orthogonal coordinate system.

In an orthogonal system of coordinates (u, v, w) with unit vectors $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_w$, a vector field $\mathbf{F}$ can be expressed as

$$\mathbf{F} = F_u\,\mathbf{e}_u + F_v\,\mathbf{e}_v + F_w\,\mathbf{e}_w,$$

and we know that the del operator can be written as

$$\boldsymbol{\nabla} = \mathbf{e}_u\,\frac{1}{h_u}\,\frac{\partial}{\partial u} + \mathbf{e}_v\,\frac{1}{h_v}\,\frac{\partial}{\partial v} + \mathbf{e}_w\,\frac{1}{h_w}\,\frac{\partial}{\partial w}.$$

It follows that the curl of $\mathbf{F}$ in the (u, v, w) coordinate system is

$$\boldsymbol{\nabla} \times \mathbf{F} = \left(\mathbf{e}_u\,\frac{1}{h_u}\,\frac{\partial}{\partial u} + \mathbf{e}_v\,\frac{1}{h_v}\,\frac{\partial}{\partial v} + \mathbf{e}_w\,\frac{1}{h_w}\,\frac{\partial}{\partial w}\right) \times \left(F_u\,\mathbf{e}_u + F_v\,\mathbf{e}_v + F_w\,\mathbf{e}_w\right).$$

This equation is correct, but is not immediately useful because there are still differentiations and vector products to carry out. We take a similar approach to that adopted earlier for divergence. The optional Appendix justifies the expressions that we use, but we just state the standard formulas here. Your task is to apply these formulas to specific vector fields.

Curl of a vector field in orthogonal coordinates

In any orthogonal right-handed system of coordinates (u, v, w) with scale factors h_u, h_v and h_w, a vector field

$$\mathbf{F} = F_u\,\mathbf{e}_u + F_v\,\mathbf{e}_v + F_w\,\mathbf{e}_w$$

has curl

$$\nabla \times \mathbf{F} = \frac{1}{J}\begin{vmatrix} h_u\,\mathbf{e}_u & h_v\,\mathbf{e}_v & h_w\,\mathbf{e}_w \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}, \tag{52}$$

where $J = h_u h_v h_w$ is the Jacobian factor.

When the determinant is expanded, partial derivative operators in the second row act on elements in the third row.

The requirement for the coordinate system to be orthogonal *and* *right-handed* implies, for example, that $\mathbf{e}_u \times \mathbf{e}_v = \mathbf{e}_w$ (rather than $-\mathbf{e}_w$). All the coordinate systems (u, v, w) used in this module are right-handed. For example, in spherical coordinates (r, θ, ϕ), we have $\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_\phi$.

It is easy to check that equation (52) works in Cartesian coordinates. In this system, all the scale factors are equal to 1, so $J = 1$. Moreover, $\mathbf{e}_x = \mathbf{i}$, $\mathbf{e}_y = \mathbf{j}$ and $\mathbf{e}_z = \mathbf{k}$, so we recover

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$

In cylindrical coordinates (r, ϕ, z), the scale factors are $h_r = 1$, $h_\phi = r$ and $h_z = 1$, so $J = r$. We therefore obtain the following result.

Curl in cylindrical coordinates

$$\nabla \times \mathbf{F} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\,\mathbf{e}_\phi & \mathbf{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ F_r & rF_\phi & F_z \end{vmatrix}. \tag{53}$$

Note that the scale factors accompany the unit vectors in row 1 *and* the components in row 3. Also, take care to include the overall factor $1/J$, i.e. $1/r$ in this case.

Example 8

The vector field $\mathbf{F} = r^2\,\mathbf{e}_\phi$ is in cylindrical coordinates. Find its curl.

Solution

The field $\mathbf{F}$ has cylindrical components $F_r = 0$, $F_\phi = r^2$, $F_z = 0$, so

$$\begin{aligned}\nabla \times \mathbf{F} &= \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\,\mathbf{e}_\phi & \mathbf{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & r^3 & 0 \end{vmatrix} \\ &= \frac{1}{r}\left(\mathbf{e}_r\,(0) - r\,\mathbf{e}_\phi\,(0) + \mathbf{e}_z\,(3r^2)\right) = 3r\,\mathbf{e}_z.\end{aligned}$$

Exercise 28

The following vector fields are in cylindrical coordinates. Find their curls.

(a) $\mathbf{F} = r^2\,\mathbf{e}_z$ (b) $\mathbf{G} = rz\,\mathbf{e}_\phi$ (c) $\mathbf{H} = rz\sin\phi\,\mathbf{e}_r$

In two dimensions, polar coordinates are similar to cylindrical coordinates, but there is no z-dependence. This means that we can get the expression for curl in polar coordinates by expanding equation (53), bearing in mind that $F_z = 0$, and F_r and F_ϕ are independent of z. Setting $F_z = 0$ gives

$$\boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\,\mathbf{e}_\phi & \mathbf{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ F_r & rF_\phi & 0 \end{vmatrix}.$$

Then, using $\partial F_r/\partial z = 0$ and $\partial F_\phi/\partial z = 0$, we get the following result.

Curl in polar coordinates

$$\boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{r}\left(\frac{\partial(rF_\phi)}{\partial r} - \frac{\partial F_r}{\partial \phi}\right)\mathbf{e}_z. \qquad (54)$$

Exercise 29

In polar coordinates, the vector field in Exercise 25 can be expressed as

$$\mathbf{F} = \frac{1}{r}\,\mathbf{e}_\phi \quad (r \neq 0).$$

Use equation (54) to confirm that the curl of this field is equal to the zero vector.

The final coordinate system that we need to consider is spherical coordinates (r, θ, ϕ). In this case, the scale factors are $h_r = 1$, $h_\theta = r$ and $h_\phi = r\sin\theta$, so $J = r^2\sin\theta$, leading to the following result.

Curl in spherical coordinates

$$\boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\,\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta\,F_\phi \end{vmatrix}. \qquad (55)$$

Exercise 30

The following vector fields are in spherical coordinates. Find their curls.

(a) $\mathbf{F} = r\,\mathbf{e}_\theta$ (b) $\mathbf{G} = r\sin\theta\,\mathbf{e}_\phi$ (c) $\mathbf{H} = r^2\,\mathbf{e}_r$

Learning outcomes

After studying this unit, you should be able to do the following.

- Define the terms scalar field and vector field.
- Interpret contour maps of scalar fields, and interpret arrow maps and field line maps of vector fields.
- Convert a scalar or vector field expressed in Cartesian coordinates into polar, cylindrical or spherical coordinates.
- Given a scalar field, calculate its gradient field in Cartesian, polar, cylindrical or spherical coordinates.
- State and apply the properties of gradient fields.
- Given a vector field, calculate its divergence in Cartesian, polar, cylindrical or spherical coordinates.
- Given a vector field, calculate its curl in Cartesian, polar, cylindrical or spherical coordinates.
- Relate arrow maps of vector fields to their divergences and curls.

Appendix: proofs of results for div and curl

This optional Appendix is neither assessable nor examinable. Its aim is to justify the general formulas for divergence and curl in orthogonal coordinates given in equations (43) and (52). Because these proofs are difficult to find elsewhere, we include them for reference purposes and general interest. However, this material is more demanding than the general level of this module, so do not be dismayed if you find it hard.

Do not study this Appendix at the expense of other units. It can be read for interest when you have the time (perhaps after completing the module).

Consider a general orthogonal coordinate system with coordinates (u, v, w), unit vectors $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_w$, and scale factors h_u, h_v and h_w. In such a system, a vector field $\mathbf{F}$ is written as

$$\mathbf{F} = F_u\,\mathbf{e}_u + F_v\,\mathbf{e}_v + F_w\,\mathbf{e}_w,$$

and the del operator is

$$\boldsymbol{\nabla} = \mathbf{e}_u\,\frac{1}{h_u}\,\frac{\partial}{\partial u} + \mathbf{e}_v\,\frac{1}{h_v}\,\frac{\partial}{\partial v} + \mathbf{e}_w\,\frac{1}{h_w}\,\frac{\partial}{\partial w}.$$

Using these expressions, we can construct the divergence and curl in the usual way:

$$\operatorname{div}\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F} \quad\text{and}\quad \mathbf{curl}\,\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{F}.$$

In practice, however, these expressions need to be unpacked. The unit vectors may vary from point to point, and the effect of partial derivative operators such as $\partial/\partial u$ on the components and the unit vectors of $\mathbf{F}$ must be worked out. The main text skipped directly to the final results, namely equations (43) and (52), but no proofs were given. The missing proofs are contained in this Appendix.

Divergence and curl in polar coordinates

Before looking at the general problem, it is helpful to consider a specific case: the expressions for the divergence and curl of a two-dimensional vector field in polar coordinates (r, ϕ). In this case, the scale factors are $h_r = 1$ and $h_\phi = r$, and the divergence is

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \frac{1}{r} \mathbf{e}_\phi \frac{\partial}{\partial \phi} \right) \cdot (F_r \, \mathbf{e}_r + F_\phi \, \mathbf{e}_\phi). \tag{56}$$

The first task is to carry out the partial differentiations. One very important fact must be understood: the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\phi$ are not fixed, but vary with position. In fact, you saw in equations (11) that

$$\mathbf{e}_r = \cos\phi \, \mathbf{i} + \sin\phi \, \mathbf{j},$$
$$\mathbf{e}_\phi = -\sin\phi \, \mathbf{i} + \cos\phi \, \mathbf{j}.$$

Partially differentiating these equations with respect to ϕ, we obtain

$$\frac{\partial \mathbf{e}_r}{\partial \phi} = -\sin\phi \, \mathbf{i} + \cos\phi \, \mathbf{j} = \mathbf{e}_\phi,$$
$$\frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\cos\phi \, \mathbf{i} - \sin\phi \, \mathbf{j} = -\mathbf{e}_r.$$

Using these results, and noting that the unit vectors do not depend on r, the derivatives that appear in equation (56) can be evaluated as follows:

$$\frac{\partial}{\partial r}(F_r \, \mathbf{e}_r + F_\phi \, \mathbf{e}_\phi) = \frac{\partial F_r}{\partial r} \mathbf{e}_r + \frac{\partial F_\phi}{\partial r} \mathbf{e}_\phi, \tag{57}$$

$$\begin{aligned} \frac{\partial}{\partial \phi}(F_r \, \mathbf{e}_r + F_\phi \, \mathbf{e}_\phi) &= \frac{\partial F_r}{\partial \phi} \mathbf{e}_r + F_r \frac{\partial \mathbf{e}_r}{\partial \phi} + \frac{\partial F_\phi}{\partial \phi} \mathbf{e}_\phi + F_\phi \frac{\partial \mathbf{e}_\phi}{\partial \phi} \\ &= \frac{\partial F_r}{\partial \phi} \mathbf{e}_r + F_r \, \mathbf{e}_\phi + \frac{\partial F_\phi}{\partial \phi} \mathbf{e}_\phi - F_\phi \, \mathbf{e}_r. \end{aligned} \tag{58}$$

To complete the evaluation of equation (56), we must first take the scalar products of equations (57) and (58) with respect to $\mathbf{e}_r$ and $\mathbf{e}_\phi/r$, and then add the results. Since $\mathbf{e}_r$ and $\mathbf{e}_\phi$ are orthogonal unit vectors, we get

$$\mathbf{e}_r \cdot \frac{\partial}{\partial r}(F_r \, \mathbf{e}_r + F_\phi \, \mathbf{e}_\phi) = \frac{\partial F_r}{\partial r},$$
$$\frac{1}{r} \mathbf{e}_\phi \cdot \frac{\partial}{\partial \phi}(F_r \, \mathbf{e}_r + F_\phi \, \mathbf{e}_\phi) = \frac{1}{r} \left(F_r + \frac{\partial F_\phi}{\partial \phi} \right).$$

Adding these results, and using the product rule of differentiation, we get

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial F_r}{\partial r} + \frac{F_r}{r} + \frac{1}{r} \frac{\partial F_\phi}{\partial \phi} = \frac{1}{r} \frac{\partial (r F_r)}{\partial r} + \frac{1}{r} \frac{\partial F_\phi}{\partial \phi},$$

which confirms equation (44), a special case of equation (43).

A similar argument can be used for curl in polar coordinates. We start from the expression

$$\nabla \times \mathbf{F} = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \frac{1}{r}\mathbf{e}_\phi \frac{\partial}{\partial \phi}\right) \times (F_r\,\mathbf{e}_r + F_\phi\,\mathbf{e}_\phi),$$

which is like equation (56), but with a vector product rather than a scalar product.

Using equations (57) and (58), we get

$$\nabla \times \mathbf{F} = \mathbf{e}_r \times \left(\frac{\partial F_r}{\partial r}\,\mathbf{e}_r + \frac{\partial F_\phi}{\partial r}\,\mathbf{e}_\phi\right) + \frac{1}{r}\mathbf{e}_\phi \times \left(\frac{\partial F_r}{\partial \phi}\,\mathbf{e}_r + F_r\,\mathbf{e}_\phi + \frac{\partial F_\phi}{\partial \phi}\,\mathbf{e}_\phi - F_\phi\,\mathbf{e}_r\right).$$

We now evaluate the vector products. The vector product of any vector with itself is equal to zero, so $\mathbf{e}_r \times \mathbf{e}_r = \mathbf{0}$ and $\mathbf{e}_\phi \times \mathbf{e}_\phi = \mathbf{0}$. Also, as shown in Figure 36, the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ (in that order) form a right-handed system, so

$$\mathbf{e}_r \times \mathbf{e}_\phi = \mathbf{e}_z \quad \text{and} \quad \mathbf{e}_\phi \times \mathbf{e}_r = -\mathbf{e}_z.$$

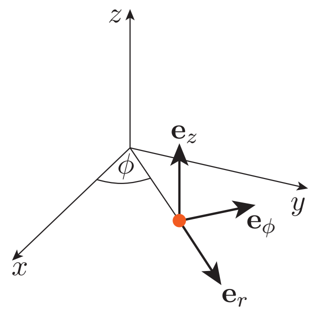


Figure 36 The unit vectors $\mathbf{e}_r$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ form a right-handed system

Multiplying out the brackets and using these results in our expression for the curl, we conclude that

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_\phi}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \phi} + \frac{F_\phi}{r}\right)\mathbf{e}_z = \frac{1}{r}\left(\frac{\partial (rF_\phi)}{\partial r} - \frac{\partial F_r}{\partial \phi}\right)\mathbf{e}_z,$$

which confirms equation (54), a special case of equation (52).

General proof of the divergence formula

We now generalise to any orthogonal coordinate system. For this purpose, it is helpful to use a slightly different notation in which we refer to coordinates u_i, with unit vectors $\mathbf{e}_i$ and scale factors h_i, where the index i can be 1, 2 or 3. The unit vectors are mutually orthogonal and right-handed so that, for example, $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$ and $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$.

In this notation the del operator and a vector field $\mathbf{F}$ can be written compactly as

$$\nabla = \sum_i \mathbf{e}_i \frac{1}{h_i}\frac{\partial}{\partial u_i} \quad \text{and} \quad \mathbf{F} = \sum_i F_i\,\mathbf{e}_i,$$

where it is understood that the sums range from $i = 1$ to $i = 3$.

The divergence of $\mathbf{F}$ in orthogonal coordinates can then be written as

$$\nabla \cdot \mathbf{F} = \left(\sum_j \mathbf{e}_j \frac{1}{h_j}\frac{\partial}{\partial u_j}\right) \cdot \left(\sum_i F_i\,\mathbf{e}_i\right). \tag{59}$$

Notice that we have used the index j in the first summation, and the index i in the second summation. This is an essential precaution. In any single summation it makes no difference whether we call the index i or j, but when we combine two summations in the same formula, we would run

into problems if we used the same index throughout. We would be in danger of leaving out terms such as

$$\mathbf{e}_1 \cdot \frac{1}{h_1} \frac{\partial}{\partial u_1}(F_2\, \mathbf{e}_2),$$

where the two indices take *different* values.

Multiplying out the brackets in equation (59) gives

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \sum_{i,j} \frac{1}{h_j}\, \mathbf{e}_j \cdot \frac{\partial (F_i\, \mathbf{e}_i)}{\partial u_j},$$

which is a sum of *nine* terms corresponding to $i = 1, 2, 3$ and $j = 1, 2, 3$.

The crucial step is the evaluation of the derivatives, especially the derivatives of the unit vectors. When we considered the case of polar coordinates, the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\phi$ were known, so their derivatives could be found explicitly. Now we need a route that works more generally, in any orthogonal coordinate system.

The key is to remember the relationship between unit vectors and tangent vectors given in equations (25) and (26). In our present notation,

$$\mathbf{e}_i = \frac{1}{h_i}\, \mathbf{T}_i, \quad \text{where } \mathbf{T}_i = \left(\frac{\partial x}{\partial u_i}, \frac{\partial y}{\partial u_i}, \frac{\partial z}{\partial u_i} \right).$$

The mixed partial derivative theorem of Unit 7 tells us that it does not matter which order is used to carry out two partial differentiations in a second-order derivative. Hence

$$\frac{\partial \mathbf{T}_i}{\partial u_j} = \frac{\partial}{\partial u_j}\left(\frac{\partial x}{\partial u_i}, \frac{\partial y}{\partial u_i}, \frac{\partial z}{\partial u_i} \right) = \frac{\partial}{\partial u_i}\left(\frac{\partial x}{\partial u_j}, \frac{\partial y}{\partial u_j}, \frac{\partial z}{\partial u_j} \right) = \frac{\partial \mathbf{T}_j}{\partial u_i}. \tag{60}$$

This equation implicitly contains all the information that we need about the spatial rates of change of the unit vectors. To take advantage of it, we write equation (59) in terms of the tangent vectors, giving

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \sum_{i,j} \frac{1}{h_j^2}\, \mathbf{T}_j \cdot \frac{\partial}{\partial u_j}\left(\frac{F_i}{h_i}\, \mathbf{T}_i \right).$$

Using the product rule of differentiation, we then have

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \sum_{i,j} \frac{1}{h_j^2} \left[\frac{\partial}{\partial u_j}\left(\frac{F_i}{h_i} \right) \mathbf{T}_j \cdot \mathbf{T}_i + \frac{F_i}{h_i}\, \mathbf{T}_j \cdot \frac{\partial \mathbf{T}_i}{\partial u_j} \right],$$

so applying equation (60), we get

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \sum_{i,j} \frac{1}{h_j^2} \left[\frac{\partial}{\partial u_j}\left(\frac{F_i}{h_i} \right) \mathbf{T}_j \cdot \mathbf{T}_i + \frac{F_i}{h_i}\, \mathbf{T}_j \cdot \frac{\partial \mathbf{T}_j}{\partial u_i} \right]. \tag{61}$$

This equation is ripe for simplification! In an orthogonal coordinate system, the tangent vectors are orthogonal, so the scalar product in the first term is equal to zero unless $j = i$, in which case it is equal to $\mathbf{T}_i \cdot \mathbf{T}_i = |\mathbf{T}_i|^2 = h_i^2$. The scalar product in the last term can be written as

$$\mathbf{T}_j \cdot \frac{\partial \mathbf{T}_j}{\partial u_i} = \frac{1}{2} \frac{\partial (\mathbf{T}_j \cdot \mathbf{T}_j)}{\partial u_i} = \frac{1}{2} \frac{\partial h_j^2}{\partial u_i} = h_j \frac{\partial h_j}{\partial u_i}.$$

Using these results in equation (61), we obtain

$$\nabla \cdot \mathbf{F} = \sum_i \left[\frac{\partial}{\partial u_i}\left(\frac{F_i}{h_i}\right) + \frac{F_i}{h_i}\left(\sum_j \frac{1}{h_j}\frac{\partial h_j}{\partial u_i}\right)\right]. \tag{62}$$

This can be tidied up by noting that

$$\sum_j \frac{1}{h_j}\frac{\partial h_j}{\partial u_i} = \frac{\partial}{\partial u_i}\sum_j \ln h_j = \frac{\partial}{\partial u_i}\ln(h_1 h_2 h_3).$$

Then, introducing the Jacobian factor $J = h_1 h_2 h_3$, we get

$$\sum_j \frac{1}{h_j}\frac{\partial h_j}{\partial u_i} = \frac{\partial}{\partial u_i}\ln J = \frac{1}{J}\frac{\partial J}{\partial u_i}.$$

Returning to equation (62) and using the product rule of differentiation, we conclude that

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \sum_i \frac{\partial}{\partial u_i}\left(\frac{F_i}{h_i}\right) + \left(\frac{F_i}{h_i}\right)\left(\frac{1}{J}\frac{\partial J}{\partial u_i}\right)\\ &= \sum_i \frac{1}{J}\frac{\partial}{\partial u_i}\left(\frac{F_i}{h_i}J\right),\end{aligned}$$

which is the required result (equation (43)).

General proof of the curl formula

We again consider orthogonal coordinates u_i, with unit vectors $\mathbf{e}_i$ and scale factors h_i (for $i = 1, 2, 3$). The coordinate system is assumed to be right-handed, so

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2. \tag{63}$$

Note the cyclic pattern based on $1 \to 2 \to 3 \to 1 \to 2 \to \ldots$.

In such a coordinate system, the curl of $\mathbf{F}$ is given by

$$\nabla \times \mathbf{F} = \left(\sum_j \mathbf{e}_j \frac{1}{h_j}\frac{\partial}{\partial u_j}\right) \times \mathbf{F} = \left(\sum_j \frac{1}{h_j}\mathbf{e}_j \times \frac{\partial \mathbf{F}}{\partial u_j}\right). \tag{64}$$

In contrast with the divergence calculation, we do not need to expand $\mathbf{F}$ in terms of its components.

Let us focus on a single component of the curl. We consider $(\nabla \times \mathbf{F})_3$, the component in the local direction of the unit vector $\mathbf{e}_3$. This is found by taking the scalar product of equation (64) with $\mathbf{e}_3$, giving

$$\begin{aligned}(\nabla \times \mathbf{F})_3 = \frac{1}{h_1}\mathbf{e}_3 \cdot \left(\mathbf{e}_1 \times \frac{\partial \mathbf{F}}{\partial u_1}\right) + \frac{1}{h_2}\mathbf{e}_3 \cdot \left(\mathbf{e}_2 \times \frac{\partial \mathbf{F}}{\partial u_2}\right)\\ + \frac{1}{h_3}\mathbf{e}_3 \cdot \left(\mathbf{e}_3 \times \frac{\partial \mathbf{F}}{\partial u_3}\right).\end{aligned}$$

You may remember that equation (37) of Unit 4 gave the identity

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c},$$

which is valid for any vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Using this identity, we get

$$(\boldsymbol{\nabla} \times \mathbf{F})_3 = \frac{1}{h_1}\,(\mathbf{e}_3 \times \mathbf{e}_1) \cdot \frac{\partial \mathbf{F}}{\partial u_1} + \frac{1}{h_2}\,(\mathbf{e}_3 \times \mathbf{e}_2) \cdot \frac{\partial \mathbf{F}}{\partial u_2} + \frac{1}{h_3}\,(\mathbf{e}_3 \times \mathbf{e}_3) \cdot \frac{\partial \mathbf{F}}{\partial u_3}.$$

The last term is equal to zero because the vector product of any vector with itself is equal to the zero vector. Equations (63) then give

$$(\boldsymbol{\nabla} \times \mathbf{F})_3 = \frac{1}{h_1}\,\mathbf{e}_2 \cdot \frac{\partial \mathbf{F}}{\partial u_1} - \frac{1}{h_2}\,\mathbf{e}_1 \cdot \frac{\partial \mathbf{F}}{\partial u_2}.$$

Expressing this in terms of the tangent vectors $\mathbf{T}_i = h_i\,\mathbf{e}_i$, we have

$$(\boldsymbol{\nabla} \times \mathbf{F})_3 = \frac{1}{h_1 h_2}\left(\mathbf{T}_2 \cdot \frac{\partial \mathbf{F}}{\partial u_1} - \mathbf{T}_1 \cdot \frac{\partial \mathbf{F}}{\partial u_2}\right). \tag{65}$$

Using the product rule, the term in brackets can be expressed as

$$\mathbf{T}_2 \cdot \frac{\partial \mathbf{F}}{\partial u_1} - \mathbf{T}_1 \cdot \frac{\partial \mathbf{F}}{\partial u_2} = \left(\frac{\partial(\mathbf{T}_2 \cdot \mathbf{F})}{\partial u_1} - \frac{\partial \mathbf{T}_2}{\partial u_1} \cdot \mathbf{F}\right) - \left(\frac{\partial(\mathbf{T}_1 \cdot \mathbf{F})}{\partial u_2} - \frac{\partial \mathbf{T}_1}{\partial u_2} \cdot \mathbf{F}\right).$$

Now, the second terms in each bracket cancel out because equation (60) ensures that $\partial \mathbf{T}_2/\partial u_1 = \partial \mathbf{T}_1/\partial u_2$. So returning to equation (65), we have

$$(\boldsymbol{\nabla} \times \mathbf{F})_3 = \frac{1}{h_1 h_2}\left(\frac{\partial(\mathbf{T}_2 \cdot \mathbf{F})}{\partial u_1} - \frac{\partial(\mathbf{T}_1 \cdot \mathbf{F})}{\partial u_2}\right).$$

Finally, noting that

$$\mathbf{T}_i \cdot \mathbf{F} = h_i\,\mathbf{e}_i \cdot \mathbf{F} = h_i F_i,$$

we conclude that

$$(\boldsymbol{\nabla} \times \mathbf{F})_3 = \frac{1}{h_1 h_2}\left(\frac{\partial(h_2 F_2)}{\partial u_1} - \frac{\partial(h_1 F_1)}{\partial u_2}\right).$$

This is equivalent to the third component of equation (52), as required. Corresponding results for the other components can be found by permuting the indices from $(1, 2, 3)$ to $(2, 3, 1)$ and $(3, 1, 2)$.

Solutions to exercises

Solution to Exercise 1

(a) Substituting $x = r\cos\phi$ and $y = r\sin\phi$, we get

$$U(r,\phi) = r^2\cos^2\phi - r^2\sin^2\phi = r^2\cos(2\phi),$$

Recall the identity $\cos 2x = \cos^2 x - \sin^2 x$.

(b) We have

$$V(r,\phi) = 2r^2\cos\phi\sin\phi = r^2\sin(2\phi).$$

Recall the identity $\sin 2x = 2\sin x\cos x$.

(c) We have

$$W(r,\phi) = (r^2\cos^2\phi + r^2\sin^2\phi)^{-1/2} = (r^2)^{-1/2} = \frac{1}{r},$$

where we have taken the positive square root because $r = \sqrt{x^2+y^2} > 0$.

Solution to Exercise 2

In cylindrical coordinates, $r^2 = x^2 + y^2$, and the field is expressed as

$$T(r,\phi,z) = 100\,e^{-(r^2+z^2)} \quad (r^2 + z^2 \leq 1).$$

The point $(r,\phi,z) = (0.5, 0, 0.5)$ gives $r^2 + z^2 = 0.25 + 0.25 = 0.5 < 1$, so it lies within the domain of the function. The value of the temperature at this point is

$$T(0.5,0,0.5) = 100\,e^{-0.5} \simeq 60.7,$$

so the temperature is $60.7°C$.

Solution to Exercise 3

(a) In cylindrical coordinates, we use the equations $x = r\cos\phi$, $y = r\sin\phi$ and $z = z$ to get

$$U(r,\phi,z) = \frac{z}{(r^2\cos^2\phi + r^2\sin^2\phi + z^2)^{1/2}} = \frac{z}{(r^2+z^2)^{1/2}}.$$

Alternatively, equation (6) can be used in the denominator of $U(x,y,z)$.

Here, r is the radial coordinate of cylindrical coordinates, which is the distance from the z-axis (not the distance from the origin).

(b) In spherical coordinates, we use the equations $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$ to get

$$\begin{aligned} U(r,\theta,\phi) &= \frac{r\cos\theta}{(r^2\sin^2\theta(\cos^2\phi+\sin^2\phi) + r^2\cos^2\theta)^{1/2}} \\ &= \frac{r\cos\theta}{(r^2\sin^2\theta + r^2\cos^2\theta)^{1/2}} \\ &= \frac{r\cos\theta}{r} \\ &= \cos\theta. \end{aligned}$$

Alternatively, equation (8) can be used in the denominator of $U(x,y,z)$.

Solution to Exercise 4

The magnitude of the vector $\mathbf{e}_r$ is

$$|\mathbf{e}_r| = \sqrt{(\cos\phi)^2 + (\sin\phi)^2} = \sqrt{\cos^2\phi + \sin^2\phi} = 1,$$

and the magnitude of the vector $\mathbf{e}_\phi$ is

$$|\mathbf{e}_\phi| = \sqrt{(-\sin\phi)^2 + (\cos\phi)^2} = \sqrt{\sin^2\phi + \cos^2\phi} = 1.$$

So $\mathbf{e}_r$ and $\mathbf{e}_\phi$ are unit vectors.

To show that $\mathbf{e}_r$ and $\mathbf{e}_\phi$ are orthogonal, we evaluate their scalar product:

$$\mathbf{e}_r \cdot \mathbf{e}_\phi = (\cos\phi)(-\sin\phi) + (\sin\phi)(\cos\phi) = 0.$$

Because the scalar product is zero, and neither $\mathbf{e}_r$ nor $\mathbf{e}_\phi$ is equal to the zero vector, the two vectors must be orthogonal.

Solution to Exercise 5

In cylindrical coordinates, the vector field takes the form

$$\mathbf{F}(r,\phi,z) = F_r\,\mathbf{e}_r + F_\phi\,\mathbf{e}_\phi + F_z\,\mathbf{e}_z,$$

where

$$\mathbf{e}_r = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}, \quad \mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}, \quad \mathbf{e}_z = \mathbf{k}.$$

Using Procedure 1, we get

$$\begin{aligned} F_r = \mathbf{e}_r \cdot \mathbf{F} &= (\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}) \cdot ((x+y)\,\mathbf{i} + (y-x)\,\mathbf{j} + 3z\,\mathbf{k}) \\ &= (x+y)\cos\phi + (y-x)\sin\phi, \\ F_\phi = \mathbf{e}_\phi \cdot \mathbf{F} &= (-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}) \cdot ((x+y)\,\mathbf{i} + (y-x)\,\mathbf{j} + 3z\,\mathbf{k}) \\ &= -(x+y)\sin\phi + (y-x)\cos\phi, \\ F_z = \mathbf{e}_z \cdot \mathbf{F} &= \mathbf{k} \cdot ((x+y)\,\mathbf{i} + (y-x)\,\mathbf{j} + 3z\,\mathbf{k}) \\ &= 3z. \end{aligned}$$

The coordinate transformation equations for cylindrical coordinates are

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z,$$

so we get

$$\begin{aligned} F_r &= r(\cos\phi + \sin\phi)\cos\phi + r(\sin\phi - \cos\phi)\sin\phi \\ &= r(\cos^2\phi + \sin^2\phi) \\ &= r, \\ F_\phi &= -r(\cos\phi + \sin\phi)\sin\phi + r(\sin\phi - \cos\phi)\cos\phi \\ &= -r(\sin^2\phi + \cos^2\phi) \\ &= -r, \\ F_z &= 3z. \end{aligned}$$

Hence in cylindrical coordinates,

$$\mathbf{F}(r,\phi,z) = r\,\mathbf{e}_r - r\,\mathbf{e}_\phi + 3z\,\mathbf{e}_z.$$

Solution to Exercise 6

In spherical coordinates, the vector field takes the form

$$\mathbf{F}(r,\theta,\phi) = F_r\,\mathbf{e}_r + F_\theta\,\mathbf{e}_\theta + F_\phi\,\mathbf{e}_\phi,$$

where

$$\begin{aligned}
\mathbf{e}_r &= \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k},\\
\mathbf{e}_\theta &= \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k},\\
\mathbf{e}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}.
\end{aligned}$$

Using Procedure 1, we get

$$\begin{aligned}
F_r &= \mathbf{e}_r \cdot \mathbf{F} = (\sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}) \cdot z\mathbf{k} = z\cos\theta,\\
F_\theta &= \mathbf{e}_\theta \cdot \mathbf{F} = (\cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k}) \cdot z\mathbf{k} = -z\sin\theta,\\
F_\phi &= \mathbf{e}_\phi \cdot \mathbf{F} = (-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}) \cdot z\mathbf{k} = 0.
\end{aligned}$$

The coordinate transformation equations for spherical coordinates are

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$

so

$$F_r = r\cos^2\theta, \quad F_\theta = -r\sin\theta\cos\theta, \quad F_\phi = 0.$$

Hence in spherical coordinates,

$$\mathbf{F}(r,\theta,\phi) = r\cos^2\theta\,\mathbf{e}_r - r\sin\theta\cos\theta\,\mathbf{e}_\theta.$$

Solution to Exercise 7

Using equations (17), the vector field is

$$\begin{aligned}
\mathbf{F} = {} & \cos\theta(\sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k})\\
& - \sin\theta(\cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k})\\
= {} & (\cos^2\theta + \sin^2\theta)\,\mathbf{k} = \mathbf{k}.
\end{aligned}$$

Solution to Exercise 8

The three tangent vectors are

$$\begin{aligned}
\mathbf{T}_r &= \frac{\partial x}{\partial r}\,\mathbf{i} + \frac{\partial y}{\partial r}\,\mathbf{j} + \frac{\partial z}{\partial r}\,\mathbf{k},\\
\mathbf{T}_\theta &= \frac{\partial x}{\partial \theta}\,\mathbf{i} + \frac{\partial y}{\partial \theta}\,\mathbf{j} + \frac{\partial z}{\partial \theta}\,\mathbf{k},\\
\mathbf{T}_\phi &= \frac{\partial x}{\partial \phi}\,\mathbf{i} + \frac{\partial y}{\partial \phi}\,\mathbf{j} + \frac{\partial z}{\partial \phi}\,\mathbf{k}.
\end{aligned}$$

In spherical coordinates we have

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$

so

$$\begin{aligned}
\mathbf{T}_r &= \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k},\\
\mathbf{T}_\theta &= r\cos\theta\cos\phi\,\mathbf{i} + r\cos\theta\sin\phi\,\mathbf{j} - r\sin\theta\,\mathbf{k},\\
\mathbf{T}_\phi &= -r\sin\theta\sin\phi\,\mathbf{i} + r\sin\theta\cos\phi\,\mathbf{j}.
\end{aligned}$$

The magnitudes of these vectors are

$$h_r = \sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta} = 1,$$
$$h_\theta = \sqrt{r^2\cos^2\theta(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta} = r,$$
$$h_\phi = \sqrt{r^2\sin^2\theta(\sin^2\phi + \cos^2\phi)} = r\sin\theta,$$

which are the familiar scale factors for spherical coordinates. Hence the required unit vectors are

$$\mathbf{e}_r = \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k},$$
$$\mathbf{e}_\theta = \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k},$$
$$\mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j},$$

in agreement with equations (17).

Solution to Exercise 9

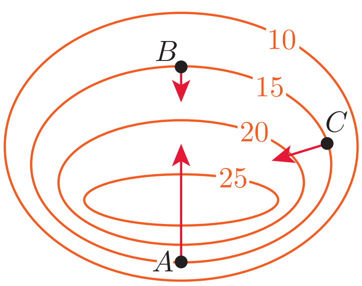
(a) With $V(x,y) = e^{x^2+y^2}$, we have

$$\begin{aligned}\mathbf{grad}\,V &= \frac{\partial V}{\partial x}\,\mathbf{i} + \frac{\partial V}{\partial y}\,\mathbf{j}\\ &= 2x\,e^{x^2+y^2}\,\mathbf{i} + 2y\,e^{x^2+y^2}\,\mathbf{j} = 2e^{x^2+y^2}\,(x\,\mathbf{i} + y\,\mathbf{j}).\end{aligned}$$

(b) With $V(x,y,z) = e^{x^2+y^2+z^2}$, we have

$$\begin{aligned}\mathbf{grad}\,V &= \frac{\partial V}{\partial x}\,\mathbf{i} + \frac{\partial V}{\partial y}\,\mathbf{j} + \frac{\partial V}{\partial z}\,\mathbf{k}\\ &= 2x\,e^{x^2+y^2+z^2}\,\mathbf{i} + 2y\,e^{x^2+y^2+z^2}\,\mathbf{j} + 2z\,e^{x^2+y^2+z^2}\,\mathbf{k}\\ &= 2e^{x^2+y^2+z^2}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}).\end{aligned}$$

Solution to Exercise 10



The required figure is shown in the margin.

Note the following points.

- The arrows representing $\mathbf{grad}\,V$ at A, B and C are perpendicular to the contour lines passing through A, B and C, respectively.
- The arrows all point 'uphill', from lower values to higher values of V.
- The arrows are longer where the contour lines are closer together; this is because the scalar field V varies more rapidly where the contour lines are closer together.

Solution to Exercise 11

To simplify the partial differentiations, we rearrange the expression for $V(x,y)$ to give

$$V(x,y) = \ln((x^2+y^2)^{1/2}) = \tfrac{1}{2}\ln(x^2+y^2).$$

Then

$$\frac{\partial V}{\partial x} = \frac{1}{2}\,\frac{1}{x^2+y^2} \times 2x = \frac{x}{x^2+y^2},$$

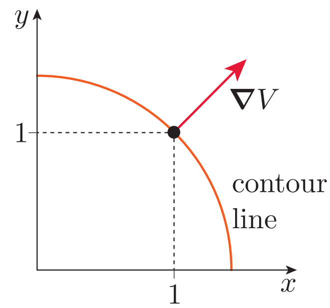
and similarly,

$$\frac{\partial V}{\partial y} = \frac{y}{x^2 + y^2}.$$

The gradient of the scalar field is therefore given by

$$\boldsymbol{\nabla} V = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2 + y^2}.$$

(This is not defined at the origin – but this is not a problem because the origin is not in the domain of the field.)



$V(x, y)$ remains constant along curves for which $x^2 + y^2$ is constant, so the contour lines are circles centred on the origin. The diagram in the margin shows the contour line through the point $(1, 1)$. At this point, $\boldsymbol{\nabla} V$ points in the direction of the radial vector $\mathbf{i} + \mathbf{j}$. An arrow representing $\boldsymbol{\nabla} V$ is shown on the diagram. This is perpendicular to the contour line, as expected.

Solution to Exercise 12

Partially differentiating $V(x, y, z)$ with respect to x gives

$$\begin{aligned}\frac{\partial V}{\partial x} &= \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{-3/2}\\ &= -\tfrac{3}{2}(x^2 + y^2 + z^2)^{-5/2} \times 2x\\ &= -\frac{3x}{(x^2 + y^2 + z^2)^{5/2}}.\end{aligned}$$

Similarly,

$$\frac{\partial V}{\partial y} = -\frac{3y}{(x^2 + y^2 + z^2)^{5/2}} \quad \text{and} \quad \frac{\partial V}{\partial z} = -\frac{3z}{(x^2 + y^2 + z^2)^{5/2}}.$$

Hence the gradient is

$$\boldsymbol{\nabla} V = -\frac{3x\,\mathbf{i} + 3y\,\mathbf{j} + 3z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{5/2}} \quad ((x, y, z) \neq (0, 0, 0)).$$

Solution to Exercise 13

(a) The gradient of the scalar field is

$$\begin{aligned}\boldsymbol{\nabla} T &= -1000 \exp(-(x^2 + 2y^2 + 2z^2)) \times (2x\,\mathbf{i} + 4y\,\mathbf{j} + 4z\,\mathbf{k})\\ &= -2000 \exp(-(x^2 + 2y^2 + 2z^2))\,(x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}).\end{aligned}$$

At the point $(1, 1, 1)$, the gradient is

$$\boldsymbol{\nabla} T\big|_{(1,1,1)} = -2000\,e^{-5}\,(\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}) \simeq -13.5\,(\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}),$$

to three significant figures, measured in degrees Celsius per metre.

(b) On moving away from $(1, 1, 1)$, the temperature increases most rapidly in the direction of the gradient at $(1, 1, 1)$. This is the direction of the vector $-(\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k})$, which has magnitude $\sqrt{1^2 + 2^2 + 2^2} = 3$, so the corresponding unit vector is

$$\widehat{\mathbf{n}} = -\tfrac{1}{3}(\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}).$$

Solution to Exercise 14

(a) The gradient is given by

$$\begin{aligned}\nabla V &= \tfrac{3}{2}(x^2+y^2+z^2)^{1/2}\,(2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k})\\ &= 3(x^2+y^2+z^2)^{1/2}\,(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}).\end{aligned}$$

So at the point $(1,2,2)$, the gradient has the value

$$\nabla V\big|_{(1,2,2)} = 9(\mathbf{i}+2\,\mathbf{j}+2\,\mathbf{k}).$$

The displacement vector from $(1,2,2)$ to $(0.98, 1.99, 2.01)$ is

$$\delta\mathbf{s} = -0.02\,\mathbf{i} - 0.01\,\mathbf{j} + 0.01\,\mathbf{k},$$

so the change in V is

$$\begin{aligned}\delta V &\simeq \nabla V \cdot \delta\mathbf{s}\\ &= 9(\mathbf{i}+2\,\mathbf{j}+2\,\mathbf{k})\cdot(-0.02\,\mathbf{i}-0.01\,\mathbf{j}+0.01\,\mathbf{k})\\ &= 9(-0.02-0.02+0.02)\\ &= -0.18.\end{aligned}$$

(b) The rate of change of V at the point $(1,2,2)$ in the direction of the unit vector $\widehat{\mathbf{n}} = (3\,\mathbf{i} + 4\,\mathbf{k})/5$ is

$$\begin{aligned}\frac{dV}{ds} &= \nabla V \cdot \widehat{\mathbf{n}}\\ &= 9(\mathbf{i}+2\,\mathbf{j}+2\,\mathbf{k})\cdot\tfrac{1}{5}(3\,\mathbf{i}+4\,\mathbf{k})\\ &= \tfrac{99}{5}\\ &= 19.8.\end{aligned}$$

Solution to Exercise 15

In spherical coordinates, the gradient is

$$\nabla V = \frac{\partial V}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\,\mathbf{e}_\phi.$$

The required partial derivatives are

$$\frac{\partial V}{\partial r} = -\frac{3}{r^4}, \quad \frac{\partial V}{\partial \theta} = 0, \quad \frac{\partial V}{\partial \phi} = 0.$$

So

$$\nabla V = -\frac{3}{r^4}\,\mathbf{e}_r.$$

This gradient field has magnitude $3/r^4$ and points radially inwards, towards the origin.

Solution to Exercise 16

In cylindrical coordinates, the gradient is

$$\nabla f = \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\,\mathbf{e}_\phi + \frac{\partial f}{\partial z}\,\mathbf{e}_z.$$

The required partial derivatives are

$$\frac{\partial f}{\partial r} = 2r\sin(2\phi), \quad \frac{\partial f}{\partial \phi} = 2r^2\cos(2\phi), \quad \frac{\partial f}{\partial z} = 2z.$$

So the gradient is

$$\boldsymbol{\nabla} f = 2r\sin(2\phi)\,\mathbf{e}_r + 2r\cos(2\phi)\,\mathbf{e}_\phi + 2z\,\mathbf{e}_z.$$

The unit vectors $\mathbf{e}_r$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ are mutually orthogonal, so the *square* of the magnitude of the gradient is given by the sum of the squares of its components:

$$|\boldsymbol{\nabla} f|^2 = 4r^2\sin^2(2\phi) + 4r^2\cos^2(2\phi) + 4z^2 = 4(r^2 + z^2).$$

Hence the magnitude of the gradient is

$$|\boldsymbol{\nabla} f| = 2\sqrt{r^2 + z^2}.$$

Solution to Exercise 17

(a) In spherical coordinates, the gradient vector field is

$$\boldsymbol{\nabla} T = \frac{\partial T}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\,\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\,\mathbf{e}_\phi.$$

The required partial derivatives are

$$\frac{\partial T}{\partial r} = \sin\theta, \quad \frac{\partial T}{\partial \theta} = r\cos\theta, \quad \frac{\partial T}{\partial \phi} = 0.$$

So

$$\boldsymbol{\nabla} T = \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta.$$

(b) The rate of change of T in the direction of the unit vector $\widehat{\mathbf{n}} = (\mathbf{e}_\theta + \mathbf{e}_\phi)/\sqrt{2}$ is

$$\begin{aligned}\widehat{\mathbf{n}} \cdot \boldsymbol{\nabla} T &= \frac{1}{\sqrt{2}}(\mathbf{e}_\theta + \mathbf{e}_\phi) \cdot (\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta) \\ &= \frac{1}{\sqrt{2}}\cos\theta.\end{aligned}$$

The vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are mutually orthogonal and of unit magnitude.

Note that the required rate of change is the component of the gradient in the direction of the unit vector $\widehat{\mathbf{n}}$, which is most readily evaluated in spherical coordinates in this case.

Solution to Exercise 18

In polar coordinates,

$$\boldsymbol{\nabla} = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\phi\frac{1}{r}\frac{\partial}{\partial \phi}.$$

In cylindrical coordinates,

$$\boldsymbol{\nabla} = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\phi\frac{1}{r}\frac{\partial}{\partial \phi} + \mathbf{e}_z\frac{\partial}{\partial z}.$$

In spherical coordinates,

$$\boldsymbol{\nabla} = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}.$$

Solution to Exercise 19

(a) $$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$
$$= 2xy + 2yz - 2yz = 2xy.$$

(b) $$\nabla \cdot \mathbf{G} = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}$$
$$= 2(x+y) + 2(y+z) + 2(x+z) = 4(x+y+z).$$

Solution to Exercise 20

(a) It makes no sense to take the divergence of a scalar field. The formula for divergence involves the components of a vector field, but a scalar field has no direction so there are no components to use.

(b) Given that $V = x^4 + y^4 + z^4$, the gradient of V is

$$\nabla V = \frac{\partial V}{\partial x}\,\mathbf{i} + \frac{\partial V}{\partial y}\,\mathbf{j} + \frac{\partial V}{\partial z}\,\mathbf{k} = 4x^3\,\mathbf{i} + 4y^3\,\mathbf{j} + 4z^3\,\mathbf{k}.$$

Hence

$$\nabla \cdot (\nabla V) = \frac{\partial}{\partial x}(4x^3) + \frac{\partial}{\partial y}(4y^3) + \frac{\partial}{\partial z}(4z^3)$$
$$= 12(x^2 + y^2 + z^2).$$

Solution to Exercise 21

(a) To partially differentiate $F_x = x/\sqrt{x^2+y^2}$ with respect to x, we use the quotient rule:

$$\frac{\partial F_x}{\partial x} = \frac{(x^2+y^2)^{1/2} - x\left(\frac{1}{2}(x^2+y^2)^{-1/2} \times 2x\right)}{x^2+y^2}$$
$$= \frac{x^2+y^2-x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^{3/2}}.$$

Because of the symmetry between F_x and F_y, a similar result is obtained for $\partial F_y/\partial y$, but with x and y interchanged. There is no z-component in this two-dimensional case, so $\partial F_z/\partial z = 0$. Hence

$$\nabla \cdot \mathbf{F} = \frac{y^2}{(x^2+y^2)^{3/2}} + \frac{x^2}{(x^2+y^2)^{3/2}} + 0$$
$$= \frac{x^2+y^2}{(x^2+y^2)^{3/2}} = \frac{1}{\sqrt{x^2+y^2}} \qquad ((x,y) \neq (0,0)).$$

(b) The divergence $\nabla \cdot \mathbf{F}$ calculated in part (a) is positive at all points (except the origin $(0,0)$, where neither $\mathbf{F}$ nor its divergence is defined). We therefore expect $\mathbf{F}$ to diverge away from a typical point, such as that marked by the blue dot in the figure provided with the question.

This interpretation is supported by examining the figure. Let us suppose that $\mathbf{F}$ represents the flow of a fluid. Compare the arrows that enter the square on sides AB and AD with those that leave the square on sides BC and CD. All these arrows have the same length, but those on sides AB and AD are more closely parallel to the sides of

the square than those on sides BC and CD. This means that less fluid is carried into the square across sides AB and AD than is carried out across sides BC and CD. So there is a net flow of fluid out of the square, as expected for a field with positive divergence. A more systematic discussion of the flow of fluids across surfaces will be given in the next unit.

Solution to Exercise 22

(a) Using equation (45) for divergence in cylindrical coordinates, and noting that $F_r = 4r^3$, $F_\phi = 0$ and $F_z = 0$, we get

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{r} \frac{\partial(4r^4)}{\partial r} = 16r^2.$$

(b) Similarly, with $G_r = r^2 \sin\phi$, $G_\phi = 0$ and $G_z = z^2$, we get

$$\boldsymbol{\nabla} \cdot \mathbf{G} = \frac{1}{r} \frac{\partial(r^3 \sin\phi)}{\partial r} + \frac{\partial(z^2)}{\partial z} = 3r\sin\phi + 2z.$$

Solution to Exercise 23

(a) Using equation (46) for divergence in spherical coordinates, with $F_r = 4r^3$, $F_\phi = 0$ and $F_z = 0$, we get

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial(4r^5)}{\partial r} = 20r^2.$$

Note that the vector field $\mathbf{F}$ is different to that in Exercise 22(a) because the unit vector $\mathbf{e}_r$ in spherical coordinates is not the same as the unit vector $\mathbf{e}_r$ in cylindrical coordinates. It is therefore not surprising that the divergences of these fields are different.

(b) Similarly, with $G_r = 0$, $G_\theta = r\sin^2\theta$ and $G_\phi = r\cos\theta\cos\phi$, we get

$$\begin{aligned}\boldsymbol{\nabla} \cdot \mathbf{G} &= \frac{1}{r\sin\theta} \frac{\partial(r\sin^3\theta)}{\partial\theta} + \frac{1}{r\sin\theta} \frac{\partial(r\cos\theta\cos\phi)}{\partial\phi}\\ &= \frac{3r\sin^2\theta\cos\theta}{r\sin\theta} - \frac{r\cos\theta\sin\phi}{r\sin\theta}\\ &= 3\sin\theta\cos\theta - \cot\theta\sin\phi.\end{aligned}$$

Solution to Exercise 24

Using the expression for divergence in spherical coordinates, and the given fact that $\operatorname{div}\mathbf{F} = 0$ at all points except the origin, we get

$$\frac{1}{r^2} \frac{\partial(r^2 f(r))}{\partial r} = 0 \quad \text{for } r > 0.$$

So

$$\frac{\partial(r^2 f(r))}{\partial r} = 0.$$

Integrating both sides of this equation, we conclude that $r^2 f(r) = C$, where C is a constant, so $f(r) = C/r^2$, which is proportional to $1/r^2$.

Solution to Exercise 25

We have

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{V} &= \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right)\mathbf{k}\\ &= \left(\frac{(x^2+y^2)\times 1 - x\times 2x}{(x^2+y^2)^2} - \frac{(x^2+y^2)\times(-1)-(-y)\times 2y}{(x^2+y^2)^2}\right)\mathbf{k}\\ &= \left(\frac{(y^2-x^2)-(y^2-x^2)}{(x^2+y^2)^2}\right)\mathbf{k} = \mathbf{0} \quad ((x,y)\neq(0,0)).\end{aligned}$$

Solution to Exercise 26

(a)
$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ z-y & x-z & y-x\end{vmatrix}.$$

Expanding the determinant, we obtain

The minus sign in front of $\mathbf{j}$ comes from the rule for expanding a 3×3 determinant.

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F} &= \mathbf{i}\left(\frac{\partial(y-x)}{\partial y} - \frac{\partial(x-z)}{\partial z}\right) - \mathbf{j}\left(\frac{\partial(y-x)}{\partial x} - \frac{\partial(z-y)}{\partial z}\right)\\ &\quad + \mathbf{k}\left(\frac{\partial(x-z)}{\partial x} - \frac{\partial(z-y)}{\partial y}\right)\\ &= \mathbf{i}\,(1-(-1)) - \mathbf{j}\,(-1-1) + \mathbf{k}\,(1-(-1))\\ &= 2\,\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}.\end{aligned}$$

(b)
$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{G} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ zy^2 & xz^2 & yx^2\end{vmatrix}\\ &= \mathbf{i}\left(\frac{\partial(yx^2)}{\partial y} - \frac{\partial(xz^2)}{\partial z}\right) - \mathbf{j}\left(\frac{\partial(yx^2)}{\partial x} - \frac{\partial(zy^2)}{\partial z}\right)\\ &\quad + \mathbf{k}\left(\frac{\partial(xz^2)}{\partial x} - \frac{\partial(zy^2)}{\partial y}\right)\\ &= \mathbf{i}\,(x^2-2xz) - \mathbf{j}\,(2xy-y^2) + \mathbf{k}\,(z^2-2yz).\end{aligned}$$

So

$$\boldsymbol{\nabla}\times\mathbf{G} = x(x-2z)\,\mathbf{i} + y(y-2x)\,\mathbf{j} + z(z-2y)\,\mathbf{k}.$$

Solution to Exercise 27

The required curl is

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}U = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{\partial U}{\partial x} & \dfrac{\partial U}{\partial y} & \dfrac{\partial U}{\partial z}\end{vmatrix}.$$

This evaluates to

$$\nabla \times \nabla U = \mathbf{i}\left(\frac{\partial^2 U}{\partial y\,\partial z} - \frac{\partial^2 U}{\partial z\,\partial y}\right) - \mathbf{j}\left(\frac{\partial^2 U}{\partial x\,\partial z} - \frac{\partial^2 U}{\partial z\,\partial x}\right) + \mathbf{k}\left(\frac{\partial^2 U}{\partial x\,\partial y} - \frac{\partial^2 U}{\partial y\,\partial x}\right).$$

Within each of the brackets on the right-hand side, the two mixed partial derivatives are equal:

$$\frac{\partial^2 U}{\partial y\,\partial z} = \frac{\partial^2 U}{\partial z\,\partial y}, \quad \frac{\partial^2 U}{\partial x\,\partial z} = \frac{\partial^2 U}{\partial z\,\partial x}, \quad \frac{\partial^2 U}{\partial x\,\partial y} = \frac{\partial^2 U}{\partial y\,\partial x}.$$

We conclude that $\nabla \times \nabla U = \mathbf{0}$ for any scalar field U.

Solution to Exercise 28

(a) The field $\mathbf{F}$ has cylindrical components $F_r = 0$, $F_\phi = 0$, $F_z = r^2$, so

$$\begin{aligned}
\nabla \times \mathbf{F} &= \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\,\mathbf{e}_\phi & \mathbf{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & 0 & r^2 \end{vmatrix} \\
&= \frac{1}{r}\big(\mathbf{e}_r\,(0) - r\,\mathbf{e}_\phi\,(2r) + \mathbf{e}_z\,(0)\big) \\
&= -2r\,\mathbf{e}_\phi.
\end{aligned}$$

(b) The field $\mathbf{G}$ has cylindrical components $G_r = 0$, $G_\phi = rz$, $G_z = 0$, so

$$\begin{aligned}
\nabla \times \mathbf{G} &= \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\,\mathbf{e}_\phi & \mathbf{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & r^2 z & 0 \end{vmatrix} \\
&= \frac{1}{r}\big(\mathbf{e}_r\,(-r^2) - r\,\mathbf{e}_\phi\,(0) + \mathbf{e}_z\,(2rz)\big) \\
&= -r\,\mathbf{e}_r + 2z\,\mathbf{e}_z.
\end{aligned}$$

(c) The field $\mathbf{H}$ has cylindrical components $H_r = rz\sin\phi$, $H_\phi = 0$, $H_z = 0$, so

$$\begin{aligned}
\nabla \times \mathbf{H} &= \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\,\mathbf{e}_\phi & \mathbf{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ rz\sin\phi & 0 & 0 \end{vmatrix} \\
&= \frac{1}{r}\big(\mathbf{e}_r\,(0) - r\,\mathbf{e}_\phi\,(-r\sin\phi) + \mathbf{e}_z\,(-rz\cos\phi)\big) \\
&= r\sin\phi\,\mathbf{e}_\phi - z\cos\phi\,\mathbf{e}_z.
\end{aligned}$$

Solution to Exercise 29

The polar components of $\mathbf{F}$ are $F_r = 0$ and $F_\phi = 1/r$, so equation (54) gives

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{F} &= \frac{1}{r}\left(\frac{\partial(1)}{\partial r} - \frac{\partial(0)}{\partial \phi}\right)\mathbf{e}_z \\ &= \mathbf{0} \quad (r \neq 0).\end{aligned}$$

Solution to Exercise 30

(a) The field $\mathbf{F}$ has spherical components $F_r = 0$, $F_\theta = r$ and $F_\phi = 0$, so equation (55) gives

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{F} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\,\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ 0 & r^2 & 0 \end{vmatrix} \\ &= \frac{1}{r^2\sin\theta}\big(\mathbf{e}_r\,(0) - r\,\mathbf{e}_\theta\,(0) + r\sin\theta\,\mathbf{e}_\phi\,(2r)\big) \\ &= 2\,\mathbf{e}_\phi.\end{aligned}$$

(b) $\mathbf{G}$ has spherical components $G_r = 0$, $G_\theta = 0$ and $G_\phi = r\sin\theta$, so

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{G} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\,\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} \\ &= \frac{1}{r^2\sin\theta}\big(\mathbf{e}_r\,(2r^2\sin\theta\cos\theta) - r\,\mathbf{e}_\theta\,(2r\sin^2\theta) + r\sin\theta\,\mathbf{e}_\phi\,(0)\big) \\ &= 2\cos\theta\,\mathbf{e}_r - 2\sin\theta\,\mathbf{e}_\theta.\end{aligned}$$

(c) $\mathbf{H}$ has spherical components $H_r = r^2$, $H_\theta = 0$ and $H_\phi = 0$, so

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{H} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\,\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ r^2 & 0 & 0 \end{vmatrix} \\ &= \mathbf{0}.\end{aligned}$$

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Figure 35: NASA.

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